جـامـعـة بنهـــا

كلية الحاسبات والذكاء الإصطناعي

Calculus

Lecture 04

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Course Syllabus

- Chapter 1: Numbers, Sets, and Functions.
- Chapter 2: Limits and Continuity.
- Chapter 3: Derivatives and Differentiation Rules.
- Chapter 4: Applications of Differentiation.
- Chapter 5: Integrals.
- Chapter 6: Techniques of Integration.
- Chapter 7: Applications of Definite Integrals.

Chapter 2 Topics

- Definition of Limit.
- Finding Limits Graphically and Numerically.
- Limit Laws.
- One-Sided Limits.
- Infinite Limits.
- Continuity.

Definition of Limit (1/8)

- The limit is one of the tools that we use to describe the behavior of a function as the values of x approach, or become **closer and closer** to, some **particular number**.

Definition of Limit (2/8)

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Definition of Limit (3/8)

$D(f) = \mathbb{R} - \{1\}$

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

we can simplify the formula by factoring the numerator and canceling common factors:

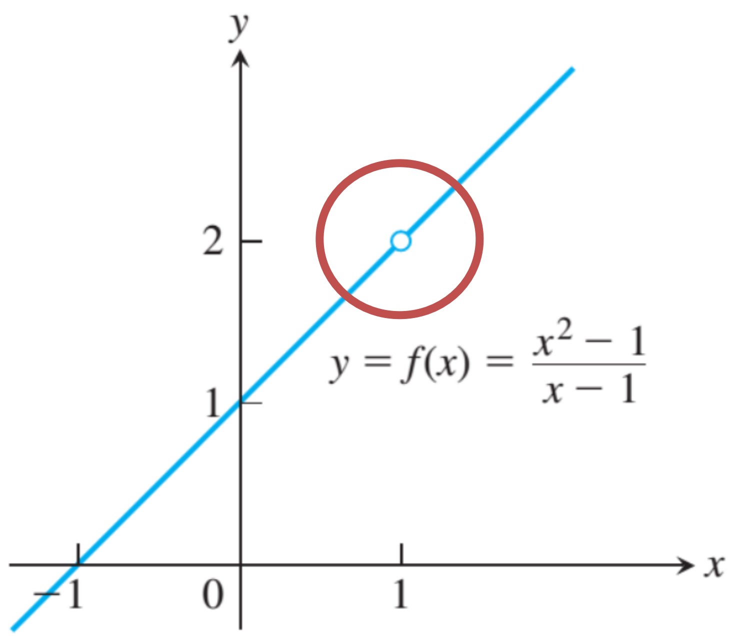
$$f(x) = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x + 1 \quad \text{for} \quad x \neq 1$$

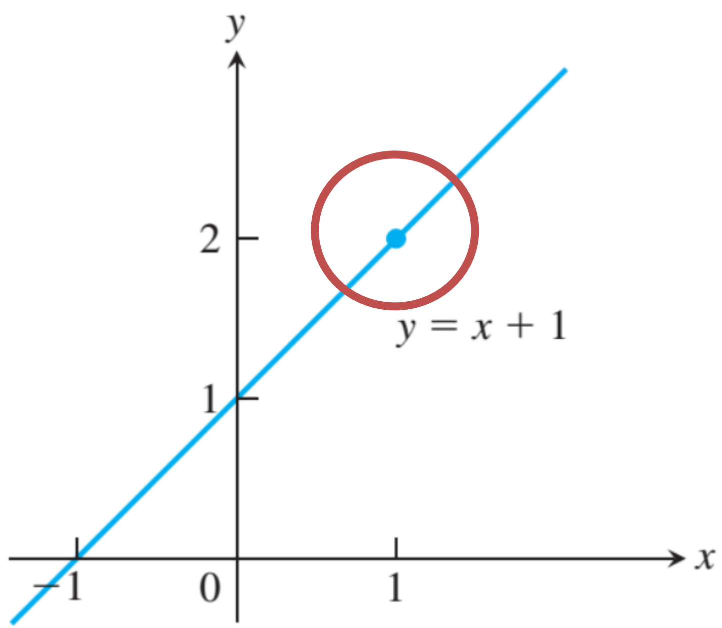
Definition of Limit (3/8)

How does the function

$$D(f) = \mathbb{R} - \{1\}$$

$$f(x) = \frac{x^2 - 1}{x - 1} = x + 1 \qquad \text{for } x \neq 1$$





Definition of Limit (4/8)

How does the function

$D(f) = \mathbb{R} - \{1\}$

$$f(x) = \frac{x^2 - 1}{x - 1} = x + 1 \qquad \text{for } x \neq 1$$

x approaches to 1 → ← x approaches to 1

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	1.9999	–	2.0001	2.001	2.01	2.1

$f(x)$ approaches to 2 → ← $f(x)$ approaches to 2

Definition of Limit (4/8)

How does the function

$$D(f) = \mathbb{R} - \{1\}$$

$$f(x) = \frac{x^2 - 1}{x - 1} = x + 1 \quad \text{for } x \neq 1$$

We would say that $f(x)$ approaches the limit 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \quad \text{or} \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Definition of Limit (5/8)

Definition:

- Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

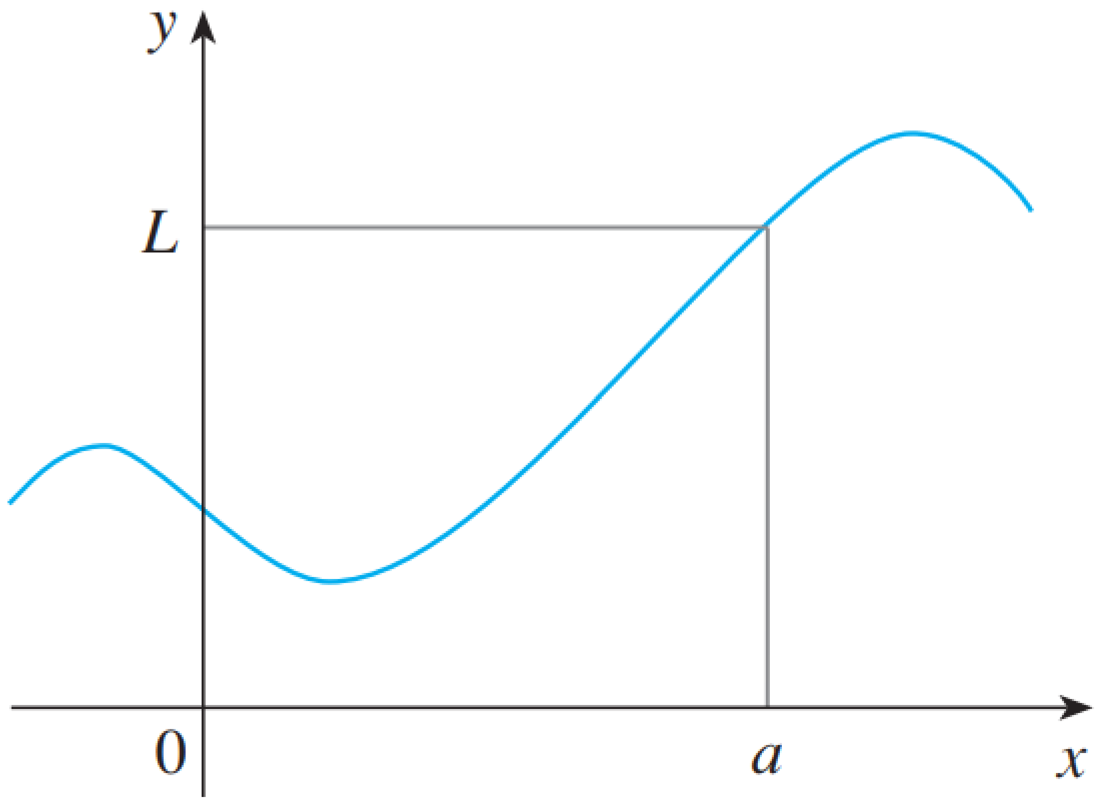
$$\lim_{x \to a} f(x) = L$$

"the limit of $f(x)$, as x approaches a, equals L"

Definition of Limit (6/8)

$$\lim_{x \to a} f(x) = L$$

in all three cases



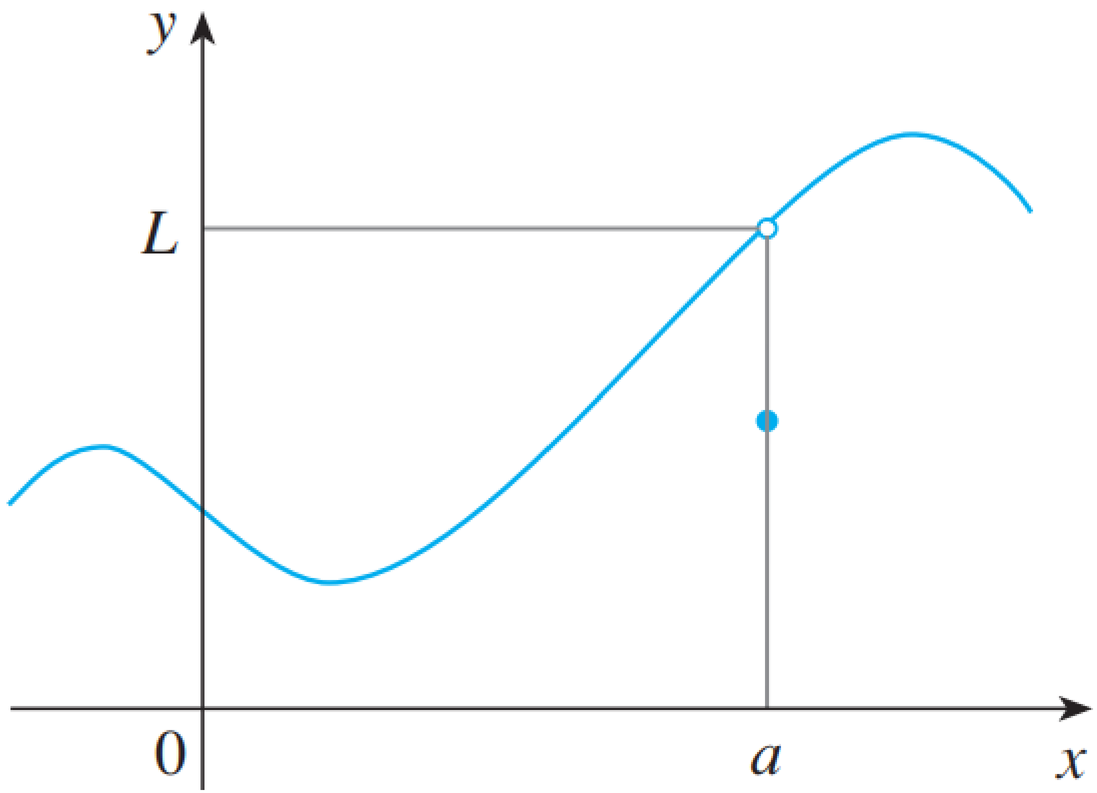
Case 1

Definition of Limit (7/8)

$$\lim_{x \to a} f(x) = L$$

in all three cases

Case 2

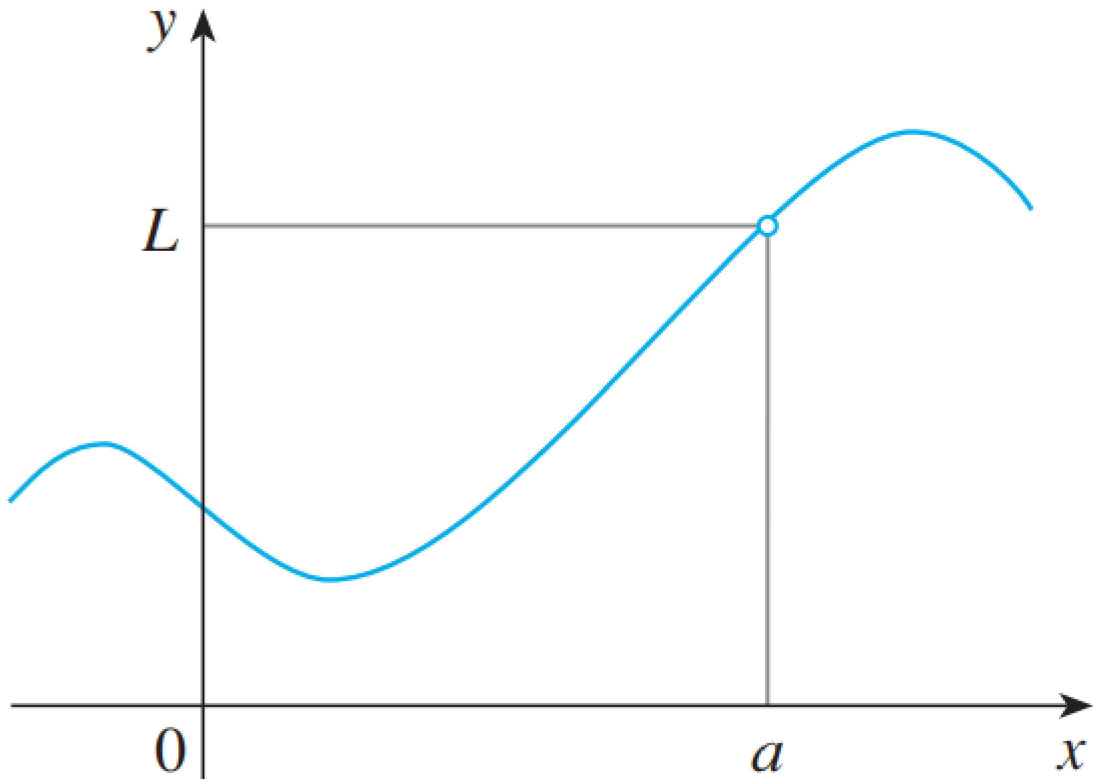


Definition of Limit (8/8)

$$\lim_{x \to a} f(x) = L$$

in all three cases

Case **3**

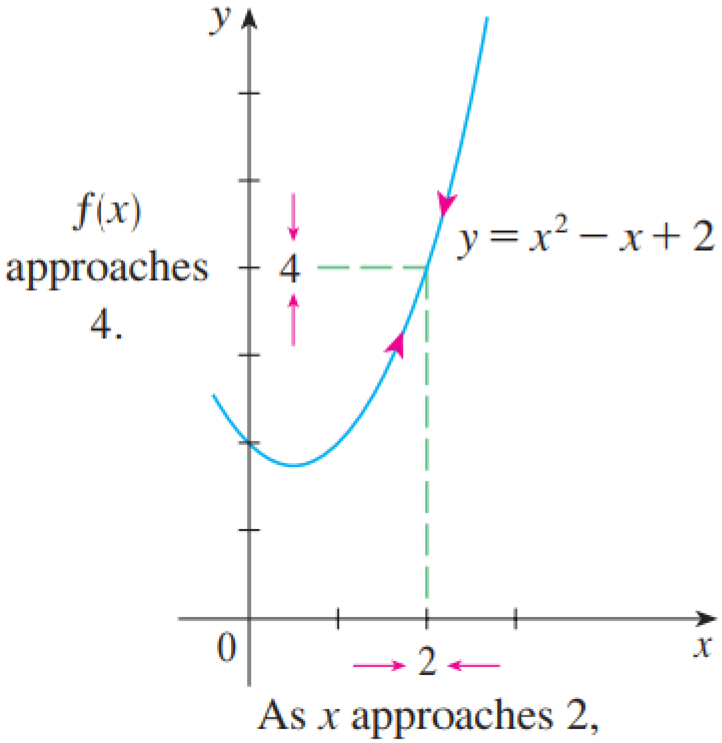


Finding Limits (1/3)

Example 1 (1/2):

- What happens to $f(x) = x^2 - x + 2$ when x is a number very close to (but not equal to) 2 ?

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

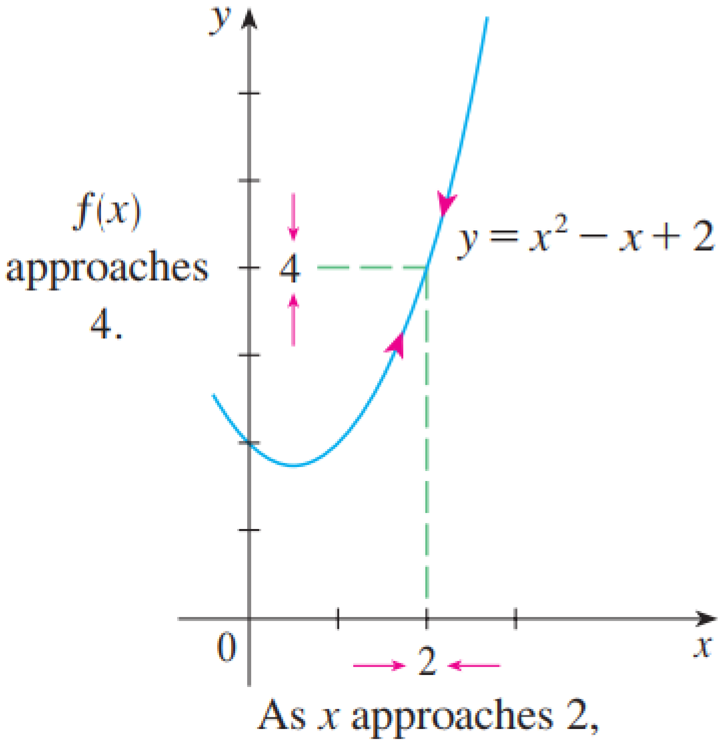


Finding Limits (1/3)

Example 1 (2/2):

- What happens to $f(x) = x^2 - x + 2$ when x is a number very close to (but not equal to) 2 ?

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$



Finding Limits (2/3)

Example 2 (1/4):

- What happens to $g(x) = \dfrac{x^3 - 2x^2}{x - 2}$ when x is 2 ?

Finding Limits (2/3)

Example 2 (2/4):

- What happens to $g(x) = \dfrac{x^3 - 2x^2}{x - 2}$ when x is 2 ?

The function $g(x)$ is undefined when $x = 2$, since the value $x = 2$ makes the denominator 0.

Finding Limits (2/3)

Example 2 (3/4):

- What happens to $g(x) = \dfrac{x^3 - 2x^2}{x - 2}$ when x is 2 ?

x approaches 2 from left → ← x approaches 2 from right

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$g(x)$	3.61	3.9601	3.996001	3.99960001		4.00040001	4.004001	4.0401	4.41

$f(x)$ approaches 4 → ↑ ← $f(x)$ approaches 4

Undefined

$$\lim_{x \to 2} g(x) = 4$$

Finding Limits (2/3)

Example 2 (4/4):

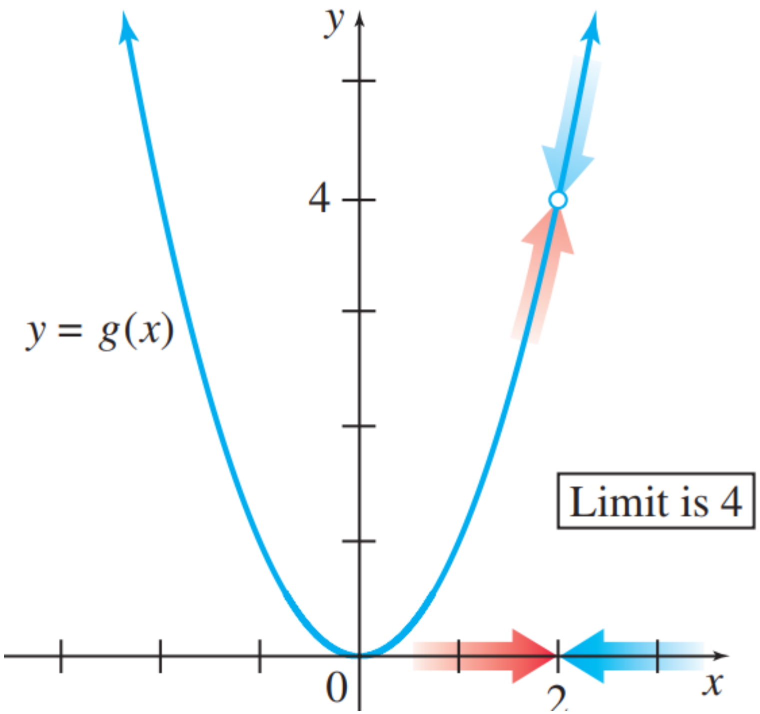
- What happens to $g(x) = \dfrac{x^3 - 2x^2}{x - 2}$ when x is 2 ?

$g(x)$ simplifies to

$$g(x) = \frac{x^2(x-2)}{x-2} = x^2,$$

provided $x \neq 2$.

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x^2 = 4.$$

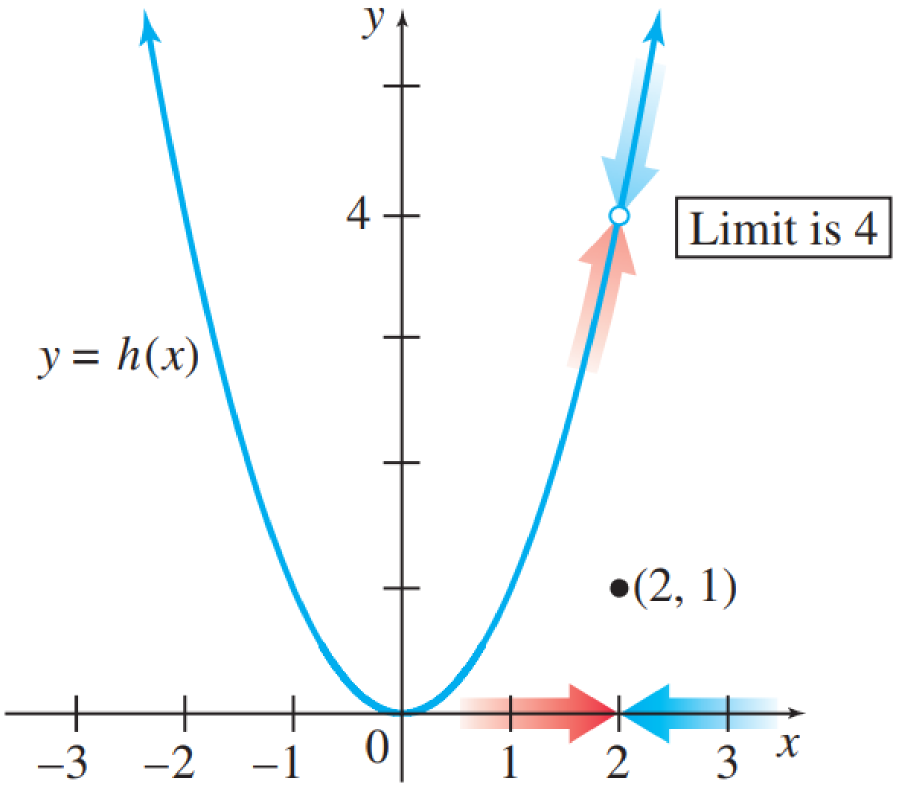


Finding Limits (3/3)

Example 3:

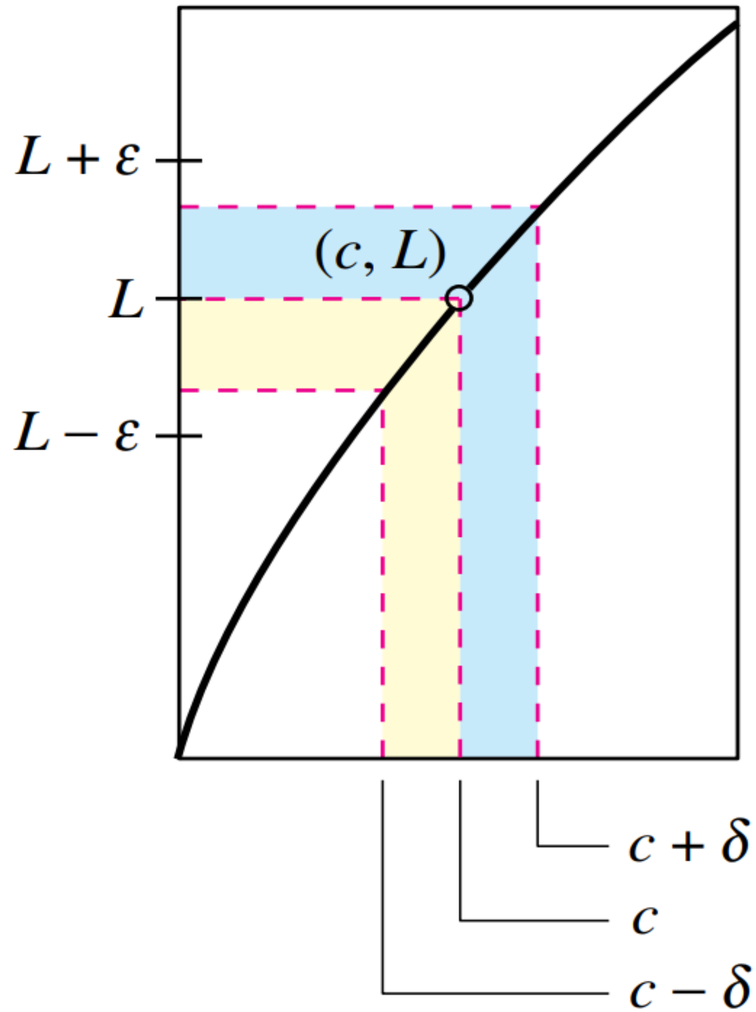
Determine $\lim_{x \to 2} h(x)$ for the function h defined by

$$h(x) = \begin{cases} x^2, & \text{if } x \neq 2, \\ 1, & \text{if } x = 2. \end{cases}$$



$$\lim_{x \to 2} h(x) = \lim_{x \to 2} x^2 = 4.$$

$\varepsilon - \delta$ Definition of Limit (1/2)



$\varepsilon - \delta$ Definition of Limit (2/2)

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

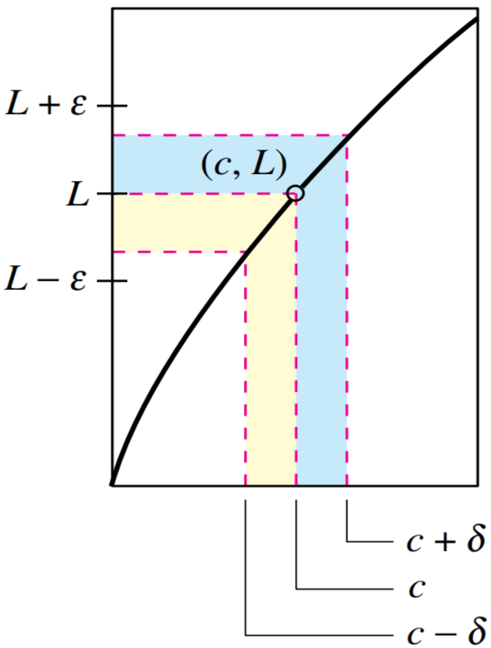
$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$



The Precise Definition of a limit

Limit Laws (1/20)

Rules for Limits

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$ exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Limit Laws (2/20)

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Limit Laws (3/20)

Example 1: Evaluate the following limit (1/2)

$$\lim_{x \to 5} (2x^2 - 3x + 4)$$

Limit Laws (3/20)

Example 1: Evaluate the following limit (2/2)

$$\lim_{x\to 5}(2x^2 - 3x + 4)$$

$$= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4$$

$$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4$$

$$= 2(5^2) - 3(5) + 4$$

$$\boxed{= 39}$$

Limit Laws (4/20)

Example 2: Evaluate the following limit (1/2)

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Example 2: Evaluate the following limit (2/2)

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)}$$

$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$

$$= -\frac{1}{11}$$

Limit Laws (5/20)

Limits of Polynomial and Rational Function

Limit of a Polynomial Function Let $p(x)$ be a polynomial function, a any real number. Then,

$$\lim_{x \to a} p(x) = p(a)$$

Limit of a Rational Function Let $r(x) = p(x)/q(x)$ be a rational function, where $p(x)$ and $q(x)$ are polynomials. Let a any real number such that $q(a) \neq 0$. Then,

$$\lim_{x \to a} r(x) = r(a)$$

Limit Laws (6/20)

Recall: Example 1: Evaluate the following limit

$$\lim_{x \to 5} (2x^2 - 3x + 4)$$

$$= 2(5^2) - 3(5) + 4$$

$$= 39$$

Limit Laws (7/20)

Recall: Example 2: Evaluate the following limit

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$

$$= -\frac{1}{11}$$

Limit Laws (8/20)

The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all a when n is odd, and is valid for $a > 0$ when n is even.

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

Limit Laws (9/20)

Example 4: Evaluate the following limit (1/2)

$$\lim_{x \to 3} \frac{x^2 - x - 1}{\sqrt{x + 1}}$$

Limit Laws (9/20)

Example 4: Evaluate the following limit (2/2)

$$\lim_{x \to 3} \frac{x^2 - x - 1}{\sqrt{x + 1}} = \frac{\lim_{x \to 3} (x^2 - x - 1)}{\lim_{x \to 3} \sqrt{x + 1}}$$

$$= \frac{\lim_{x \to 3} (x^2 - x - 1)}{\sqrt{\lim_{x \to 3} (x + 1)}}$$

$$= \frac{3^2 - 3 - 1}{\sqrt{3 + 1}}$$

$$= \frac{5}{\sqrt{4}}$$

Limit Laws (10/20)

Limits of Trigonometric Functions

Let a be a real number in the domain of the given trigonometric function.

$$\lim_{x \to a} \sin x = \sin a \qquad \lim_{x \to a} \cos x = \cos a$$

$$\lim_{x \to a} \tan x = \tan a \qquad \lim_{x \to a} \cot x = \cot a$$

$$\lim_{x \to a} \sec x = \sec a \qquad \lim_{x \to a} \csc x = \csc a$$

Limit Laws (11/20)

Limits of Trigonometric Functions (Examples)

a. $\lim_{x \to 0} \tan x = \tan(0) = 0$

b. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

Limit Laws (12/20)

Example 5: Evaluate the following limit (1/3)

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Limit Laws (12/20)

Example 5: Evaluate the following limit (2/3)

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \quad = \frac{0}{0}$$

Undetermined Value

Undetermined (Indeterminate) values

$\frac{0}{0}$	$\frac{\pm\infty}{\pm\infty}$	$\infty - \infty$	$0(\infty)$	0^0	1^∞	∞^0

Determined values

$\infty + \infty = \infty$	$-\infty - \infty = -\infty$	$0^\infty = 0$	$0^{-\infty} = \infty$	$\infty \cdot \infty = \infty$

Limit Laws (12/20)

Example 5: Evaluate the following limit (2/3)

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} \quad = \frac{0}{0}$$

Dividing Out Technique

Limit Laws (12/20)

Example 5: Evaluate the following limit (3/3)

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Dividing Out Technique

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1)$$

$$= 1 + 1 = 2$$

Limit Laws (13/20)

Example 6: Evaluate the following limit (1/4)

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

Limit Laws (13/20)

Example 6: Evaluate the following limit (2/4)

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{x-1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x^2+x+1)$$

$$= 1^2 + 1 + 1$$

$$= 3$$

Limit Laws (13/20)

Example 6: Evaluate the following limit (3/4)

$$\lim_{x\to1}\frac{x^3-1}{x-1} \quad \boxed{=3}$$

$$= \lim_{x\to1}\frac{(x-1)(x^2+x+1)}{x-1}$$

$$= \lim_{x\to1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x\to1}(x^2+x+1)$$

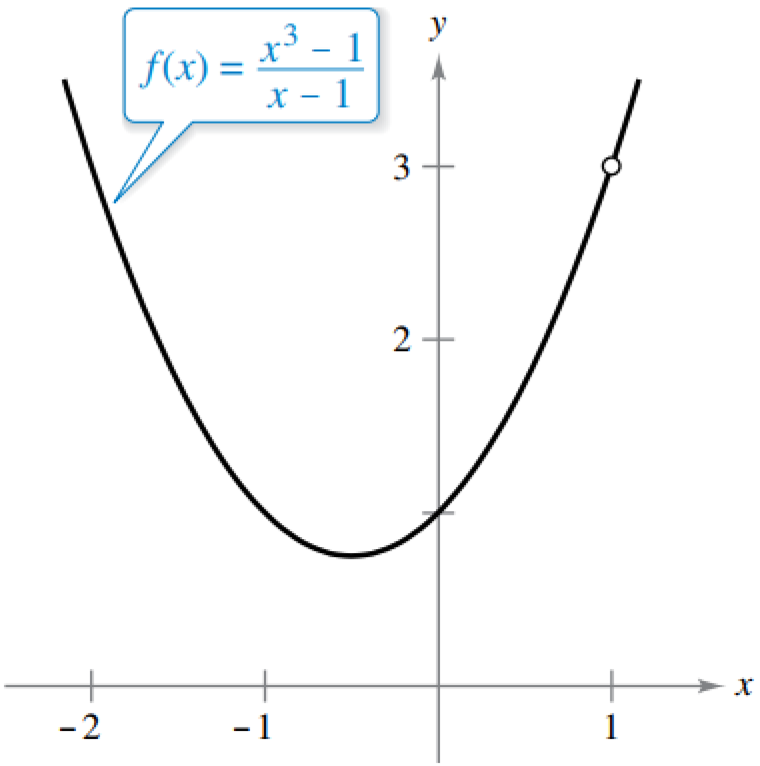
$$= 1^2+1+1$$

$$= 3$$

Limit Laws (13/20)

Example 6: Evaluate the following limit (4/4)

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$



Limit Laws (14/20)

Example 7: Evaluate the following limit (1/4)

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{0}{0}$$

Limit Laws (14/20)

Example 7: Evaluate the following limit (2/4)

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{0}{0}$$

Rationalizing Technique

Multiplying by the conjugate

Example 7: Evaluate the following limit (3/4)

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{0}{0}$$

Rationalizing Technique

Multiplying by the conjugate

$$\frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{(\sqrt{x})^2 - 2^2}{(x - 4)(\sqrt{x} + 2)}$$

$$= \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \frac{1}{\sqrt{x} + 2}$$

Limit Laws (14/20)

Example 7: Evaluate the following limit (4/4)

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{1}{4}$$

Rationalizing Technique

Multiplying by the conjugate

$$\lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

Limit Laws (15/20)

Example 8: Evaluate the following limit (1/5)

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

Limit Laws (15/20)

Example 8: Evaluate the following limit (2/5)

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \frac{0}{0}$$

Limit Laws (15/20)

Example 8: Evaluate the following limit (3/5)

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$$

$$= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$$= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}$$

$$= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}$$

$$= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0$$

Limit Laws (15/20)

Example 8: Evaluate the following limit (4/5)

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+1} + 1}$$

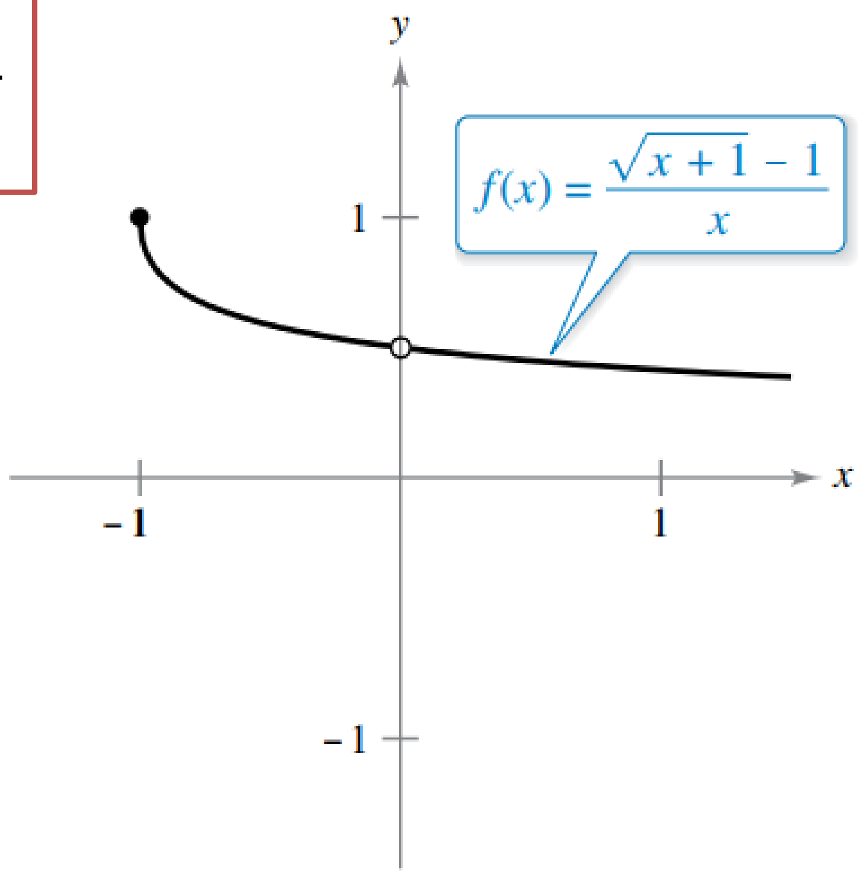
$$= \frac{1}{1+1}$$

$$= \boxed{\frac{1}{2}}$$

Limit Laws (15/20)

Example 8: Evaluate the following limit (5/5)

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \frac{1}{2}$$



Limit Laws (16/20)

The Squeeze (Sandwich/Pinching) Theorem (1/2)

If $h(x) \leq f(x) \leq g(x)$

when x is near a (except possibly at a) and

$$= \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

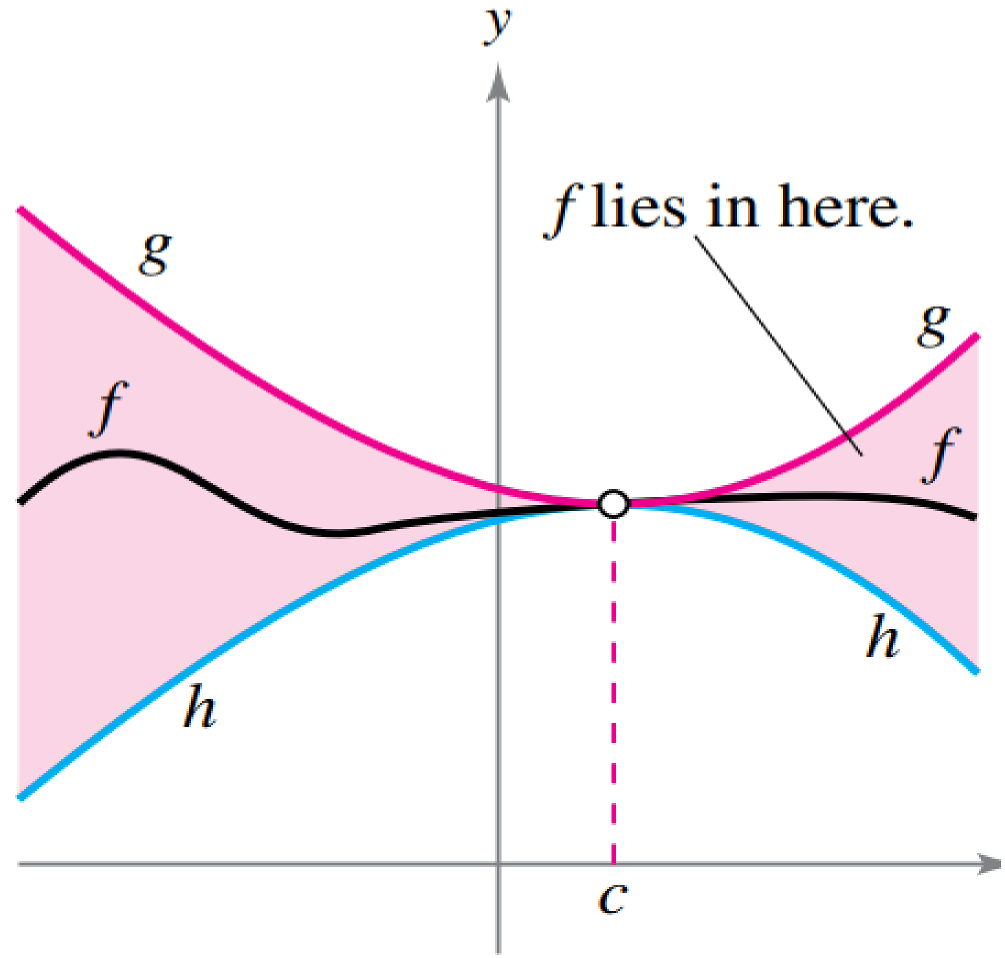
Then

$$\lim_{x \to a} f(x) = L$$

Limit Laws (16/20)

The Squeeze (Sandwich/Pinching) Theorem (2/2)

$$h(x) \leq f(x) \leq g(x)$$



Limit Laws (17/20)

Example 1:

Given that

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \textit{for all } x \neq 0$$

find $\lim_{x \to 0} u(x)$, *no matter how complicated u is.*

Limit Laws (18/20)

Example 1: $$1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2} \quad \textit{for all } x \ne 0$$

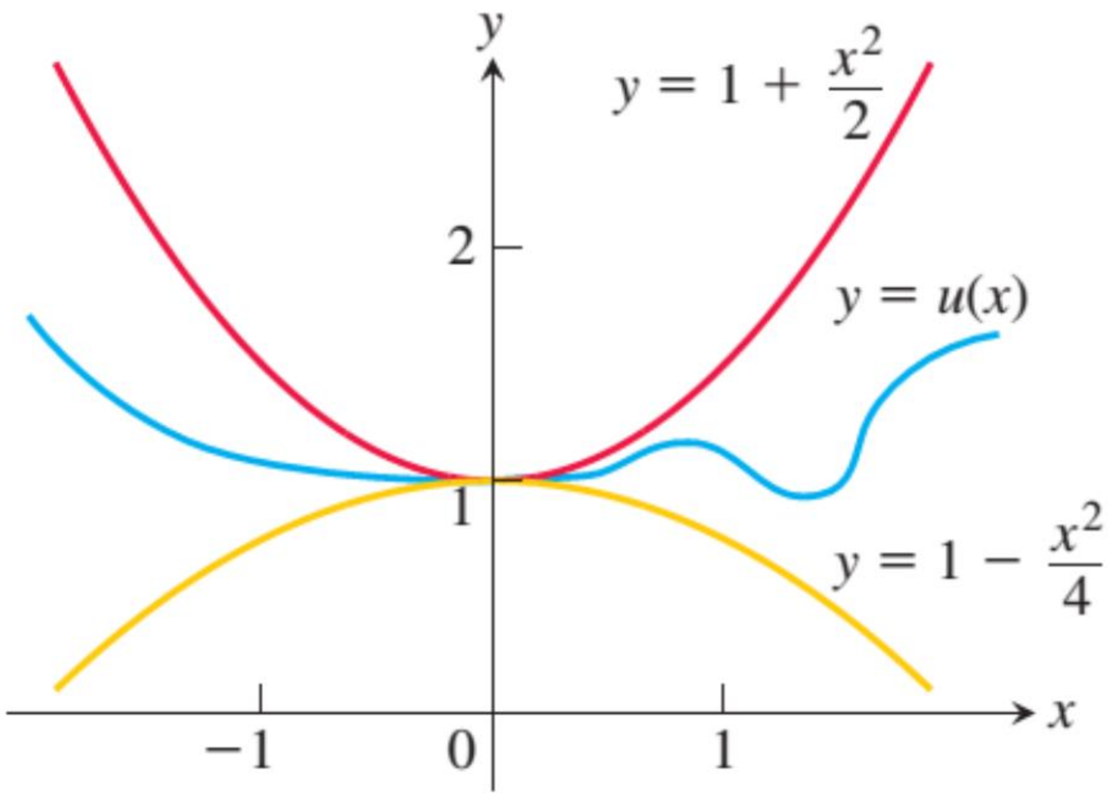
Since

$$\lim_{x\to 0}\left(1-\left(x^2/4\right)\right)=1 \quad \text{and} \quad \lim_{x\to 0}\left(1+\left(x^2/2\right)\right)=1$$

the Squeeze Theorem implies that $\lim_{x\to 0} u(x) = 1$

Limit Laws (18/20)

Example 1: $$1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2} \quad \textit{for all } x \ne 0$$



Limit Laws (19/20)

Example 2:

Using the Sandwich Theorem:

If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ *for* $-1 \le x \le 1$,

find $\lim_{x\to 0} f(x)$.

Limit Laws (20/20)

Example 2:

If $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ *for* $-1 \leq x \leq 1$,

Since

$$\lim_{x\to 0} \sqrt{5-2x^2} = \sqrt{5-2(0)^2} = \sqrt{5}$$

$$\lim_{x\to 0} \sqrt{5-x^2} = \sqrt{5-(0)^2} = \sqrt{5},$$

then by the sandwich theorem, $\lim_{x\to 0} f(x) = \sqrt{5}$.

One-Sided Limits (1/6)

Definitions (1/4):

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$as x approaches a from the **left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a.

One-Sided Limits (1/6)

Definitions (2/4):

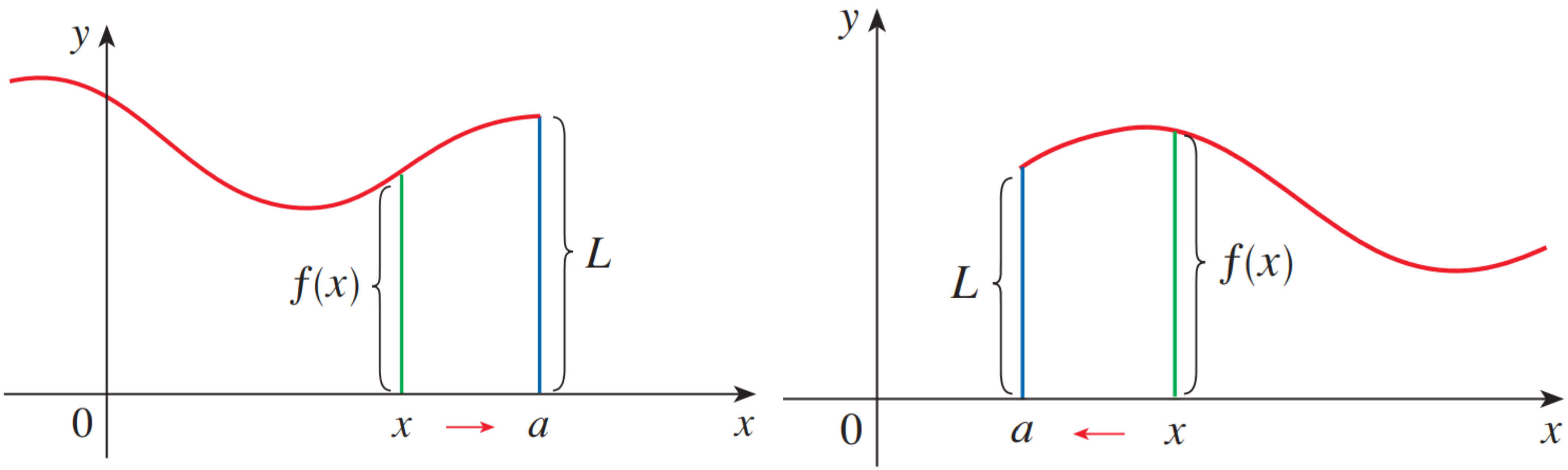
We write

$$\lim_{x \to a^+} f(x) = L$$

and say the **right-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$as x approaches a from the **right**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x greater than a.

One-Sided Limits (1/6)

Definitions (3/4):



(a) $\lim_{x \to a^-} f(x) = L$

(b) $\lim_{x \to a^+} f(x) = L$

One-Sided Limits (1/6)

Definitions (4/4):

We see that

$$\lim_{x \to a} f(x) = L \quad \textbf{if and only if}$$

$$\lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L$$

One-Sided Limits (2/6)

Example 1:

Show that $\lim_{x\to 0} f(x)$ is exist, where $f(x) = |x|$.

One-Sided Limits (2/6)

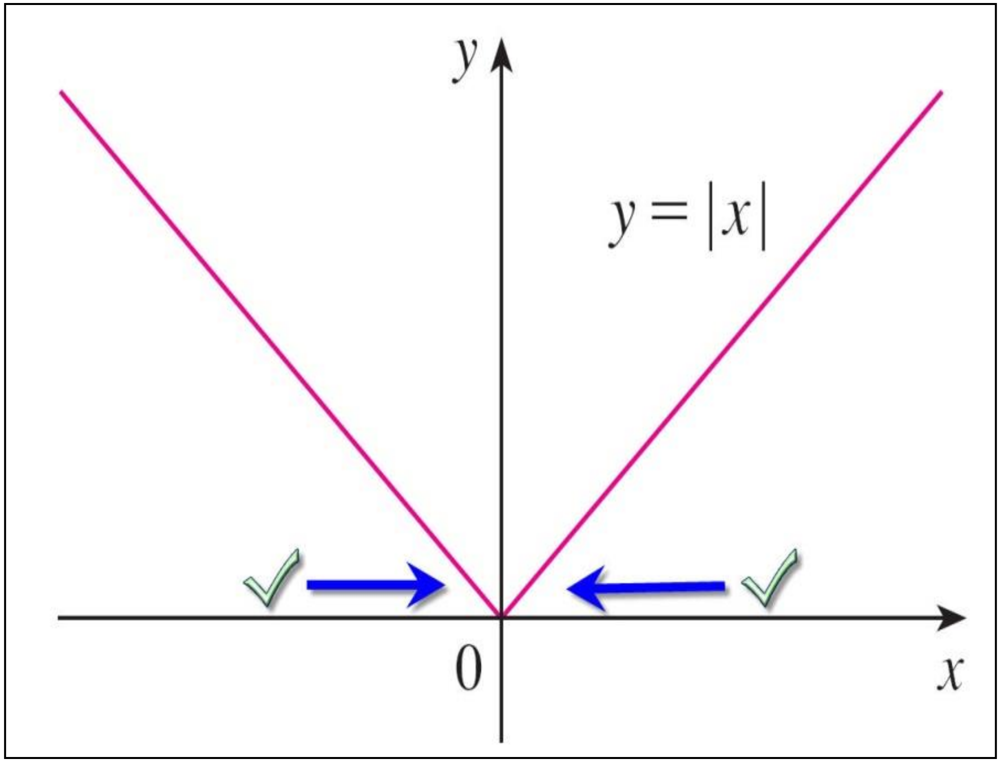
Example 1:

Show that $\lim_{x \to 0} f(x)$ is exist, where $f(x) = |x|$.

$\lim_{x \to 0^-} f(x) = 0$ and

$\lim_{x \to 0^+} f(x) = 0$ then

$\lim_{x \to 0} f(x) = 0$



One-Sided Limits (3/6)

Example 2:

Show that $\lim_{x \to 0} f(x)$ is doesn't exist, where

$$f(x) = \begin{cases} 0\,, & x < 0 \\ 1\,, & x \geq 0 \end{cases}$$

One-Sided Limits (3/6)

Example 2:

Show that $\lim_{x \to 0} f(x)$ is doesn't exist, where

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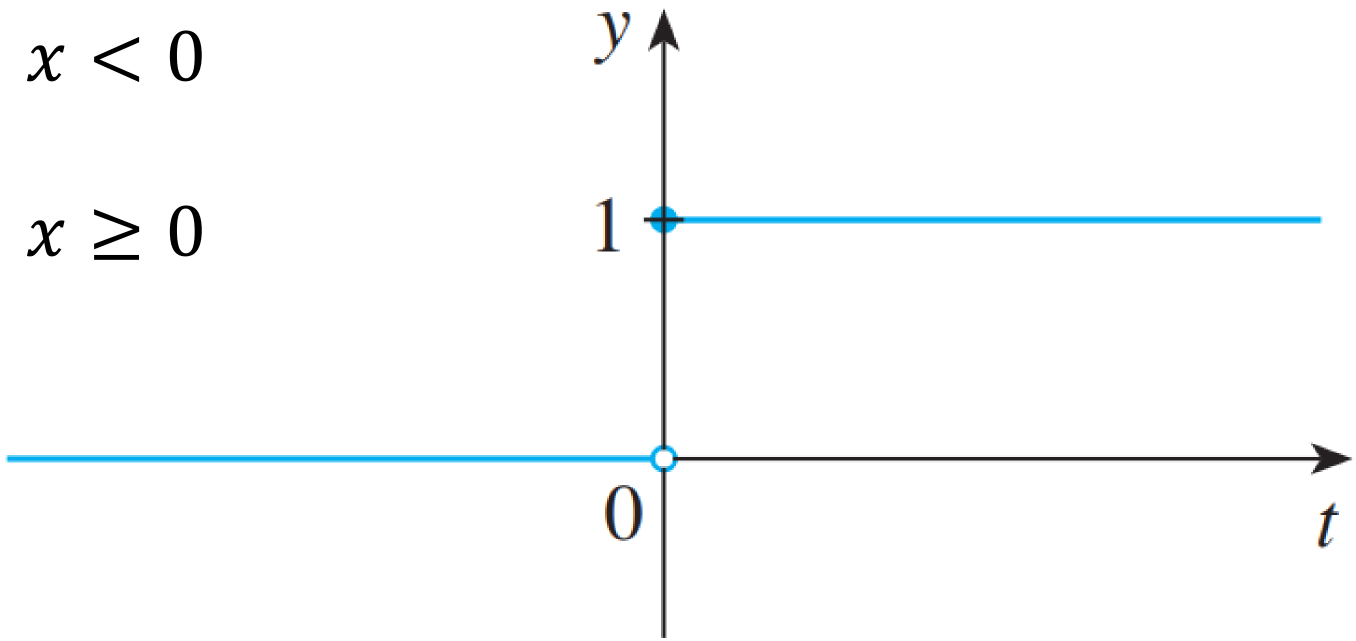
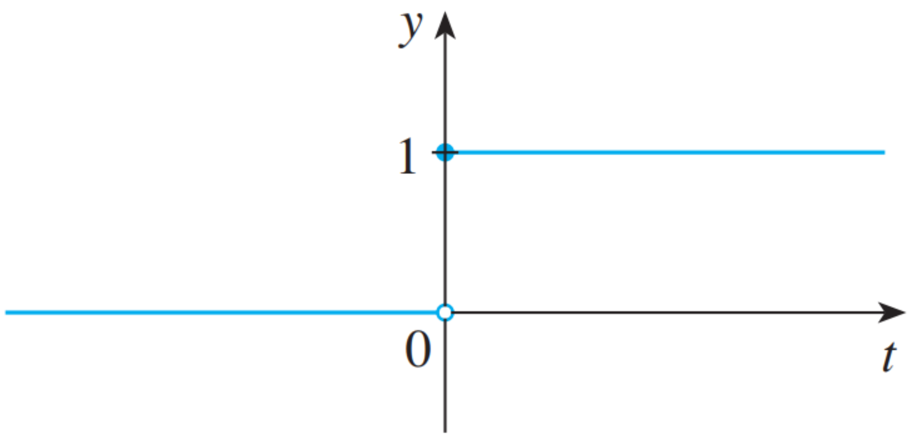


One-Sided Limits (3/6)

Example 2:

Show that $\lim_{x\to 0} f(x)$ is doesn't exist, where

$$f(x)=\begin{cases} 0\,, & x<0 \\ 1\,, & x\geq 0 \end{cases}$$



$\lim_{x\to 0^-} f(x)=0$ and

$\lim_{x\to 0^+} f(x)=1$ then

$\lim_{x\to 0} f(x)$ is doesn't exist

One-Sided Limits (4/6)

Example 3:

Show that $\lim_{x \to 0} f(x)$ is doesn't exist, where $f(x) = \dfrac{|x|}{x}$

One-Sided Limits (4/6)

Example 3:

Show that $\lim_{x\to 0} f(x)$ is doesn't exist, where $f(x) = \dfrac{|x|}{x}$

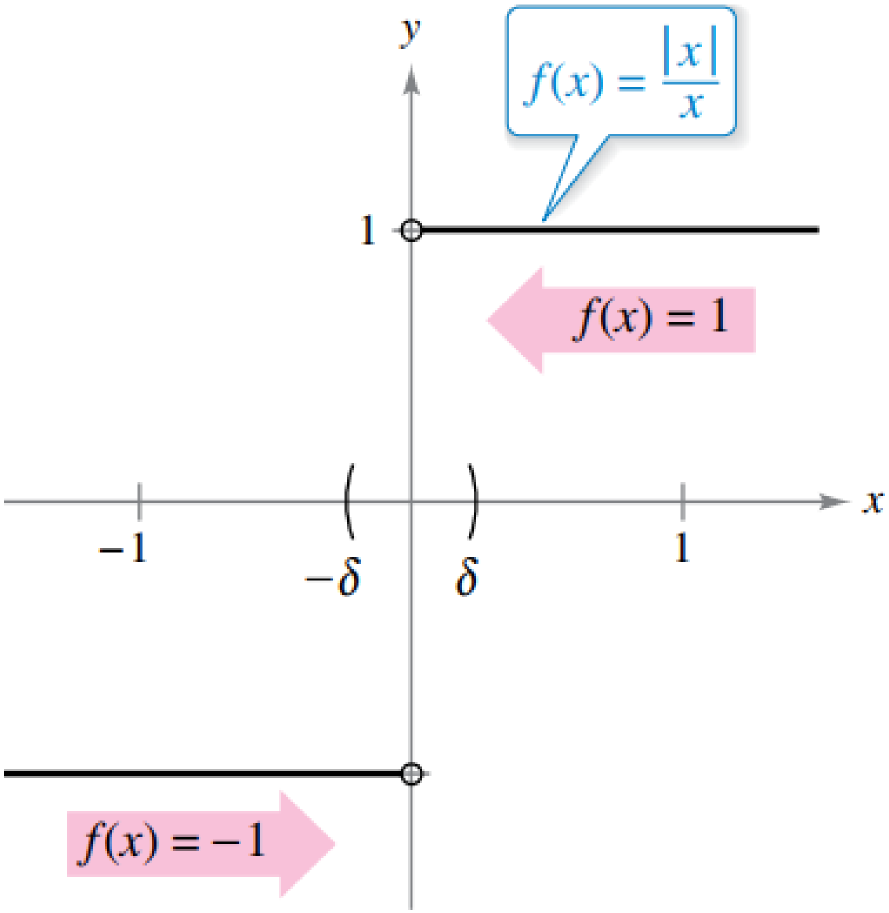
$$f(x) = \begin{cases} 1\,, & x > 0 \\ -1\,, & x < 0 \end{cases}$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

One-Sided Limits (4/6)

Example 3:

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



One-Sided Limits (4/6)

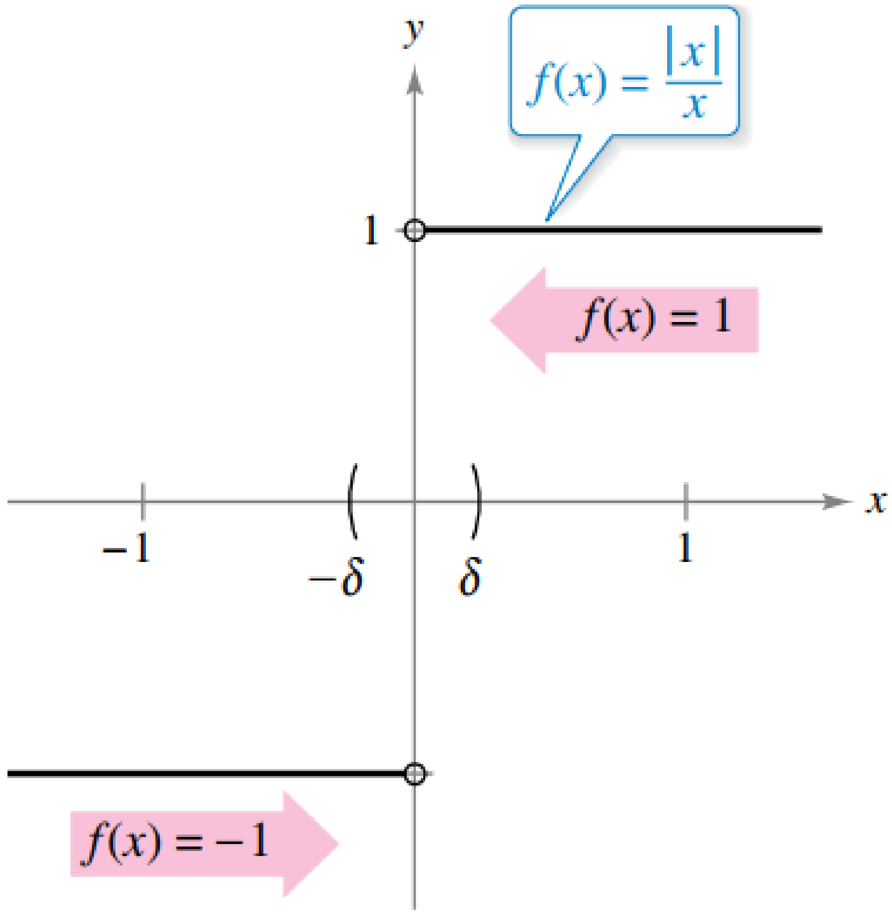
Example 3:

$$f(x) = \begin{cases} 1\,, & x > 0 \\ -1\,, & x < 0 \end{cases}$$

$\lim\limits_{x \to 0^-} f(x) = -1$ and

$\lim\limits_{x \to 0^+} f(x) = 1$ then

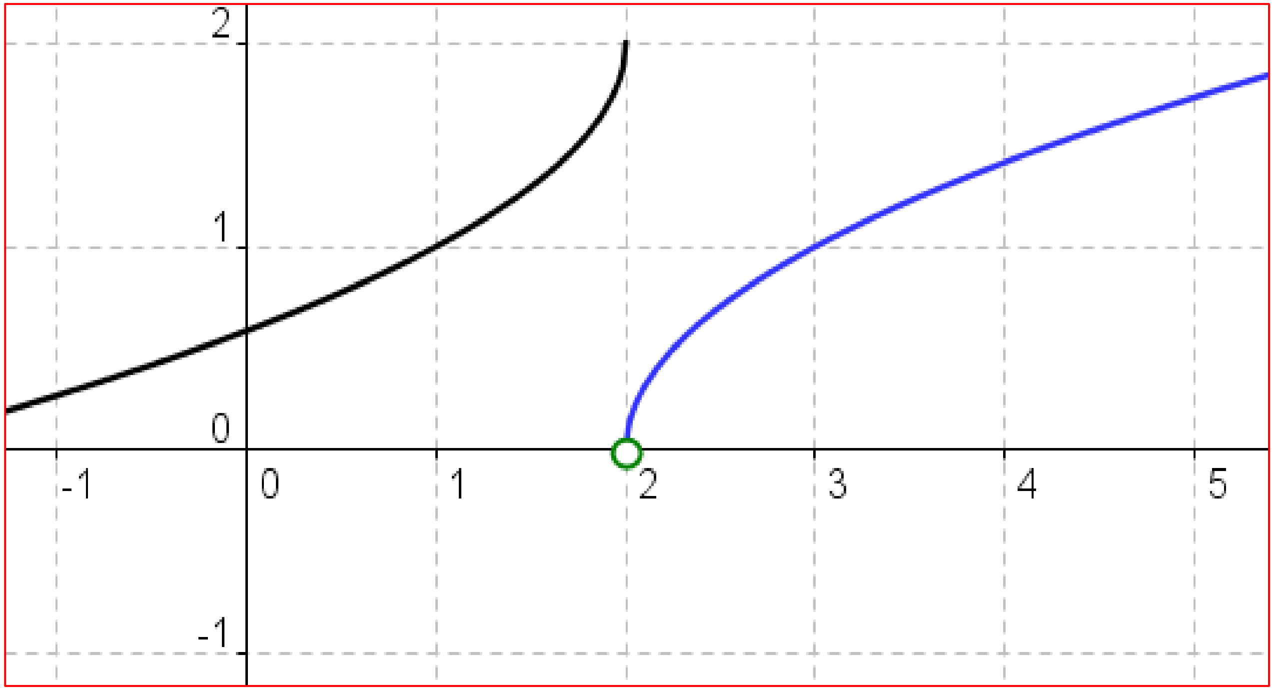
$\lim\limits_{x \to 0} f(x)$ is doesn't exist



One-Sided Limits (5/6)

Example 4:

For the function f graphed in the accompanying figure,



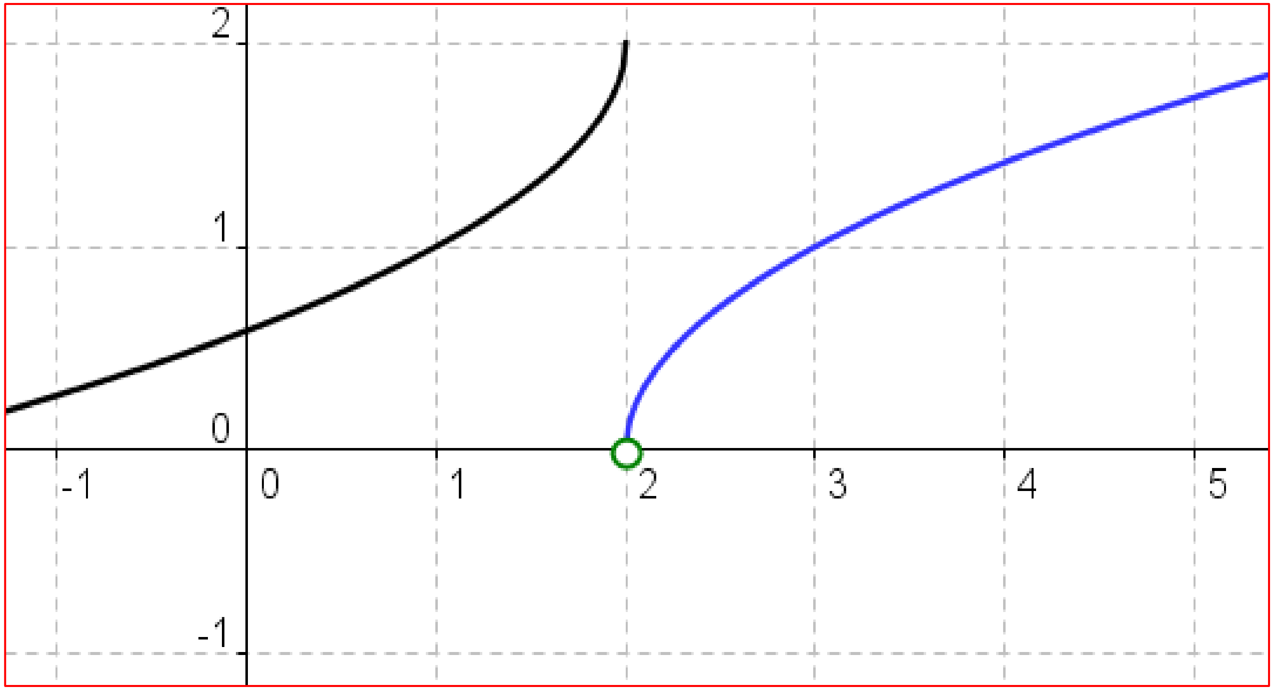
Find

a) $\lim_{x \to 2^-} f(x)$

One-Sided Limits (5/6)

Example 4:

For the function f graphed in the accompanying figure,



Find

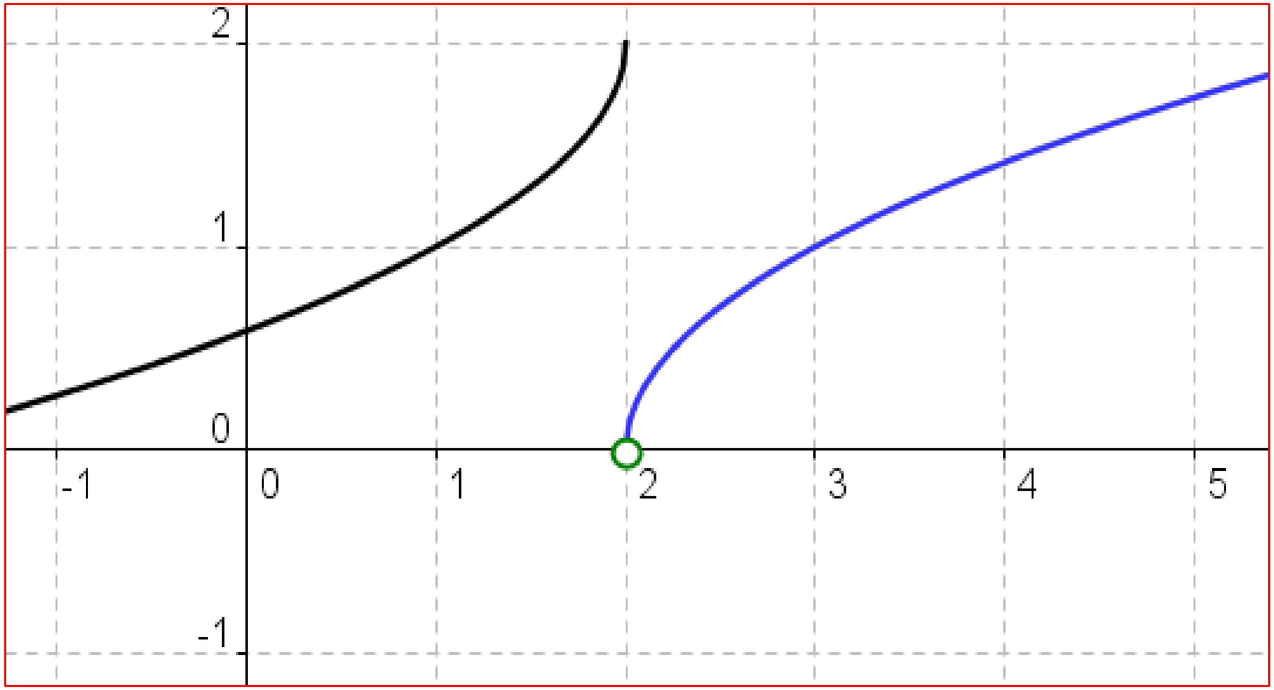
a) $\lim_{x \to 2^-} f(x)$

$= 2$

One-Sided Limits (5/6)

Example 4:

For the function f graphed in the accompanying figure,



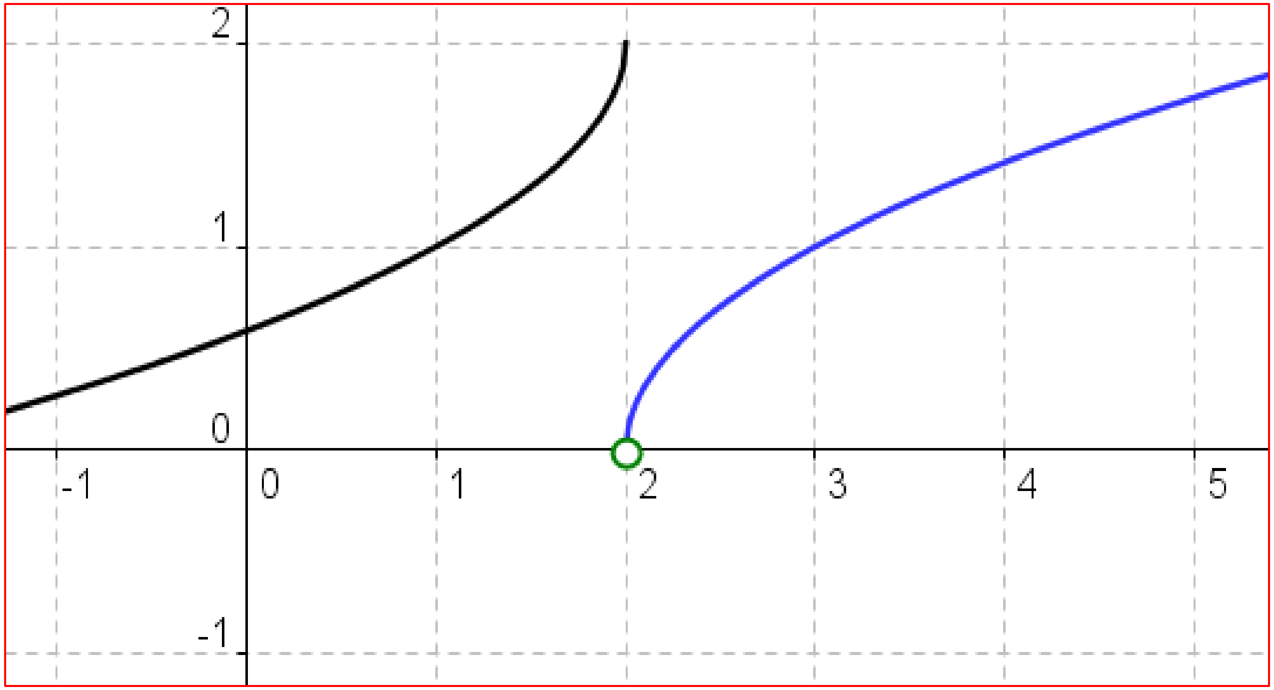
Find

b) $\lim_{x\to 2^+} f(x)$

One-Sided Limits (5/6)

Example 4:

For the function f graphed in the accompanying figure,



Find

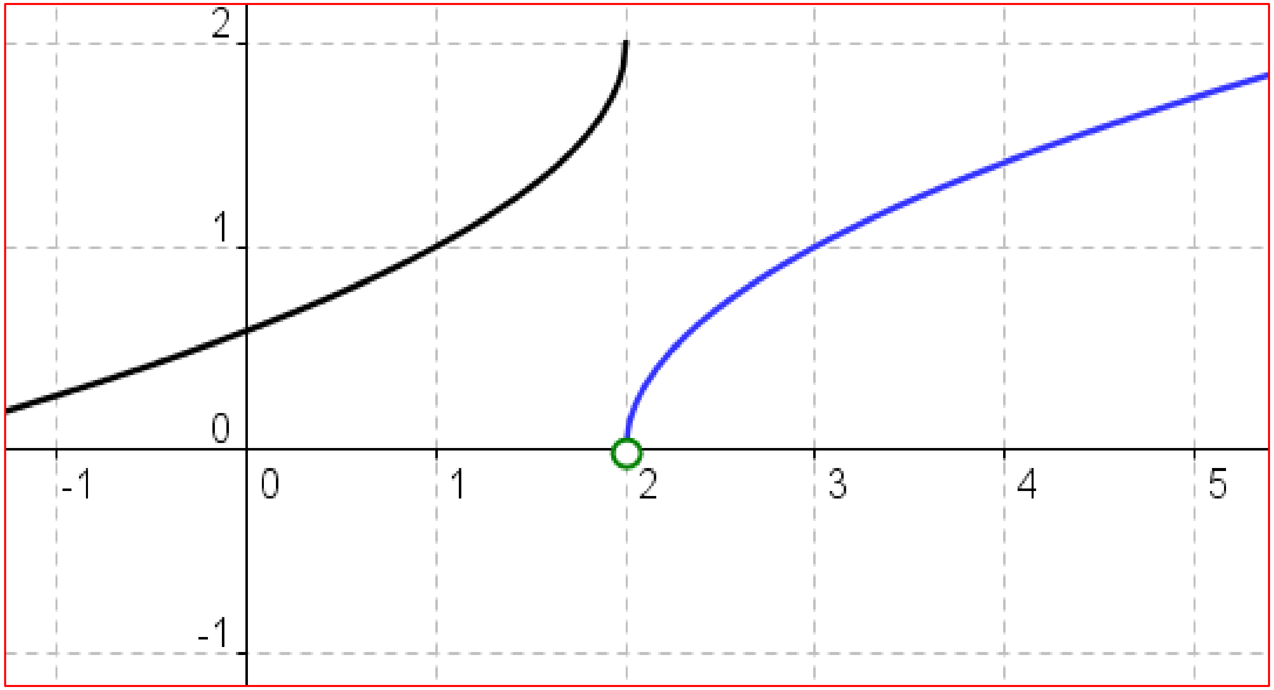
b) $\lim_{x \to 2^+} f(x)$

$= 0$

One-Sided Limits (5/6)

Example 4:

For the function f graphed in the accompanying figure,



Find

c) $\lim_{x \to 2} f(x)$

One-Sided Limits (5/6)

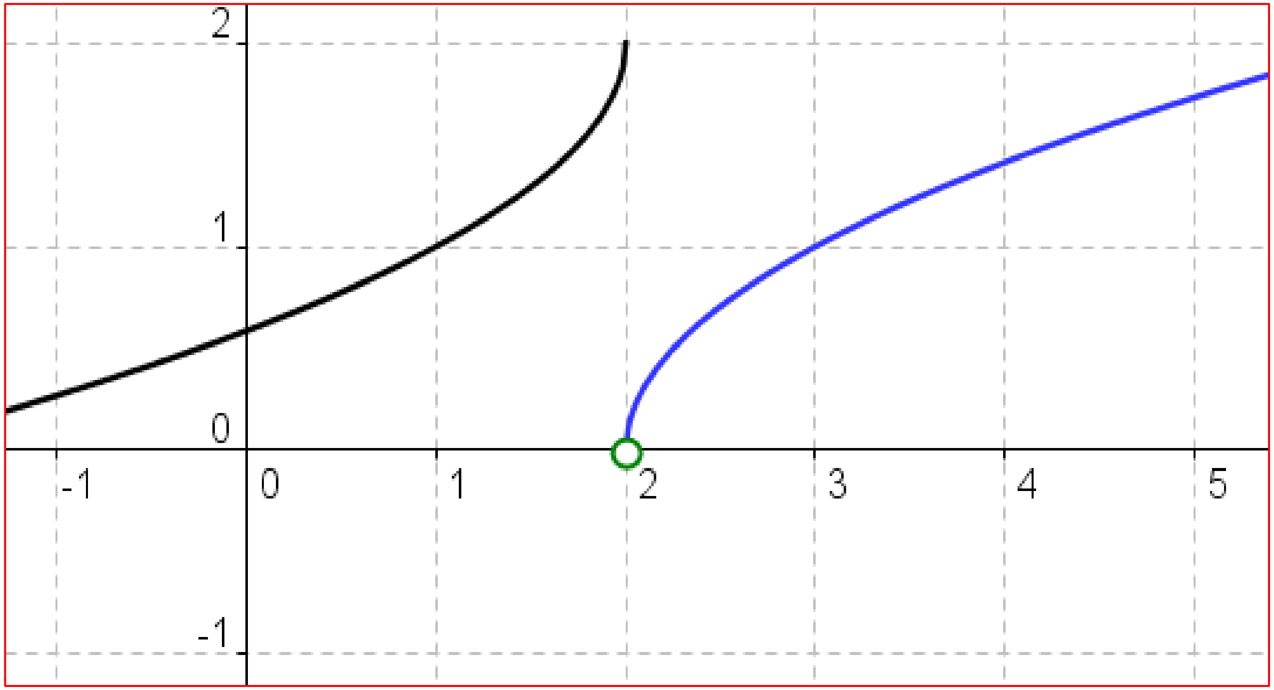
Example 4:

For the function f graphed in the accompanying figure,

<u>Find</u>

c) $\lim_{x \to 2} f(x)$

$=$ doesn't exist



One-Sided Limits (5/6)

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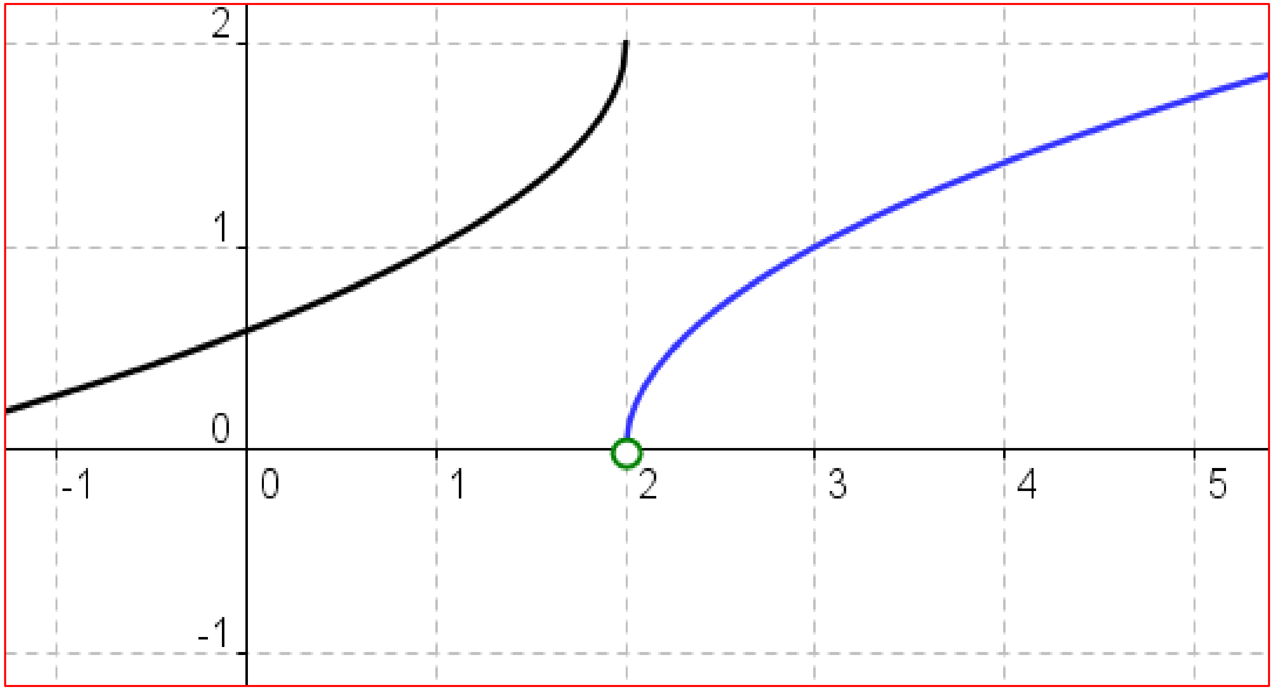


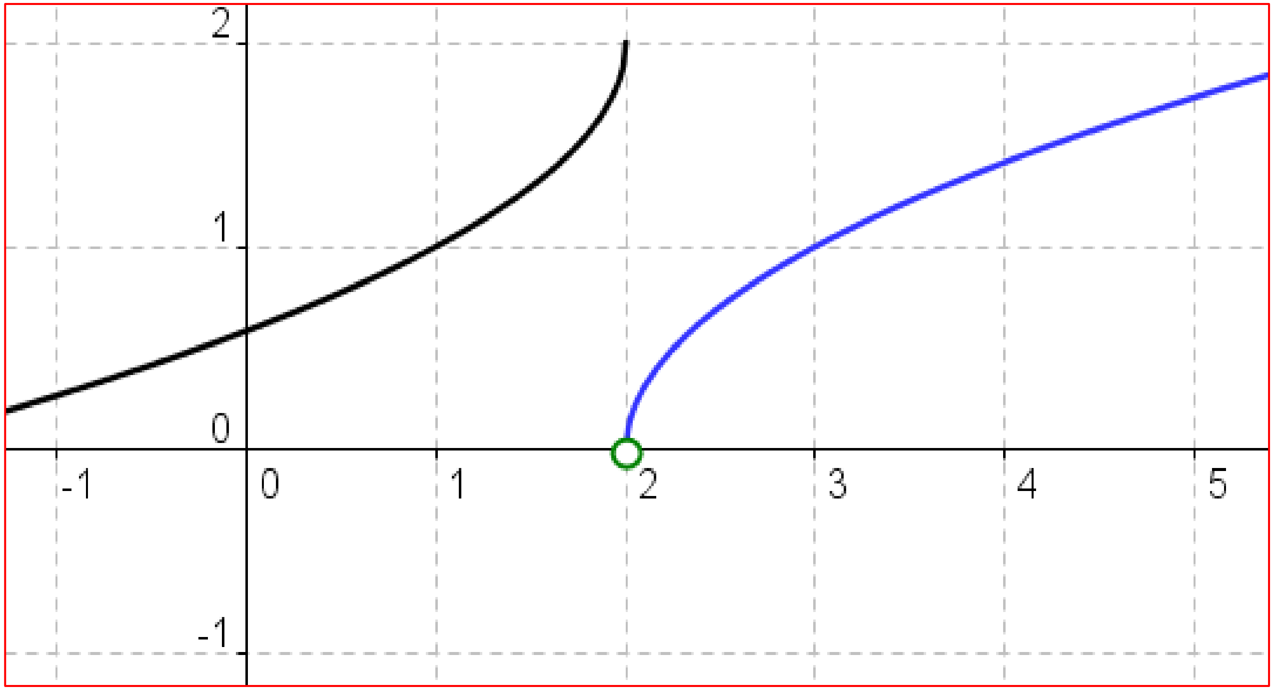
<u>Find</u>

d) $f(2)$

One-Sided Limits (5/6)

Example 4:

For the function f graphed in the accompanying figure,



<u>Find</u>

d) $f(2)$

$= 2$

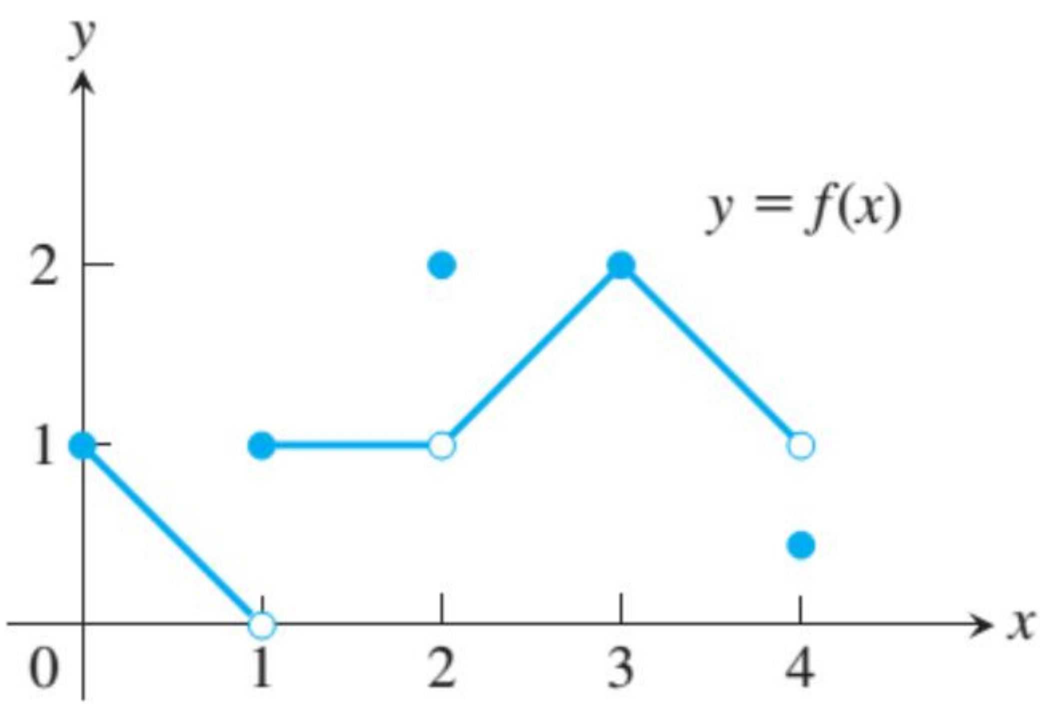
One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,

Find

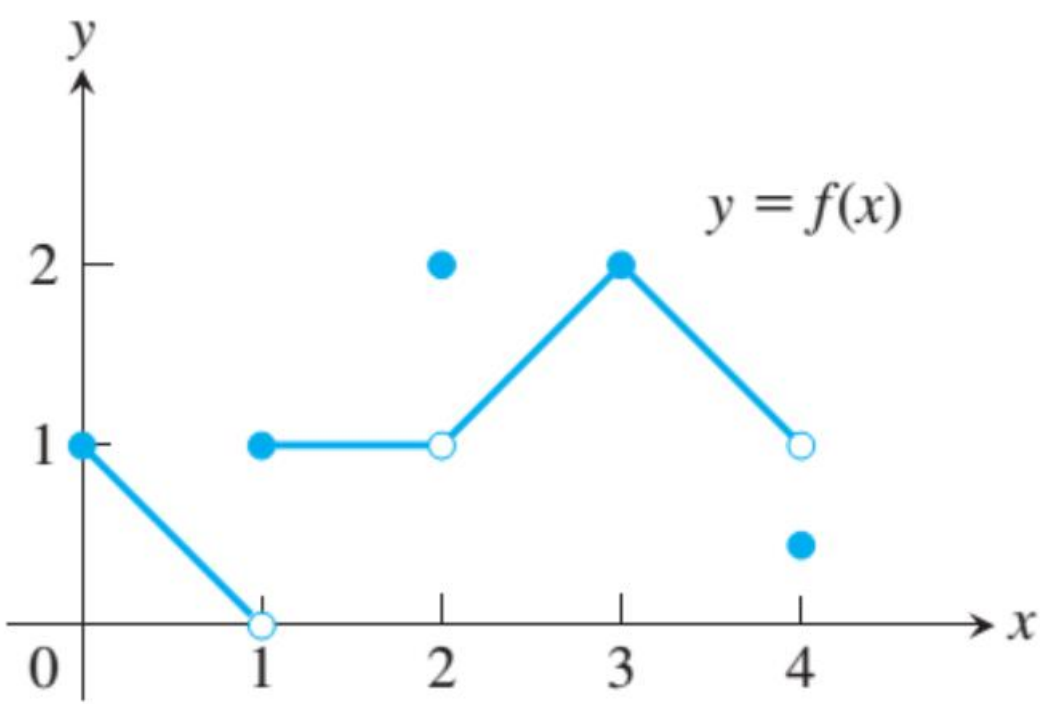
a) $\lim_{x\to 0^+} f(x)$



One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,



Find

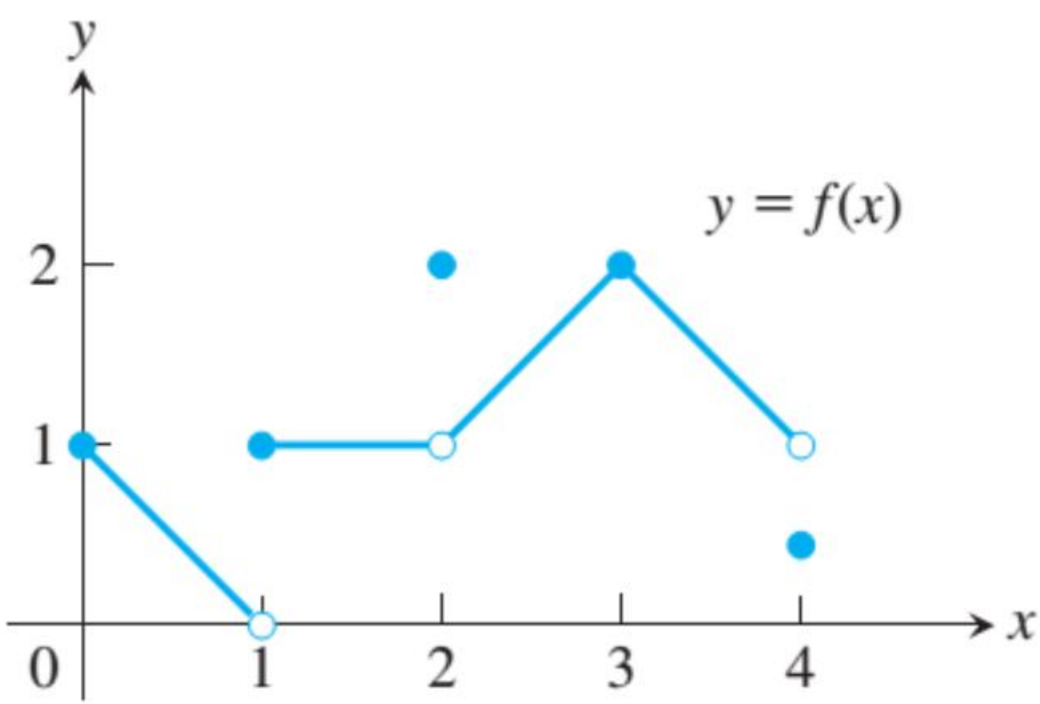
a) $\lim_{x \to 0^+} f(x)$

$= 1$

One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,



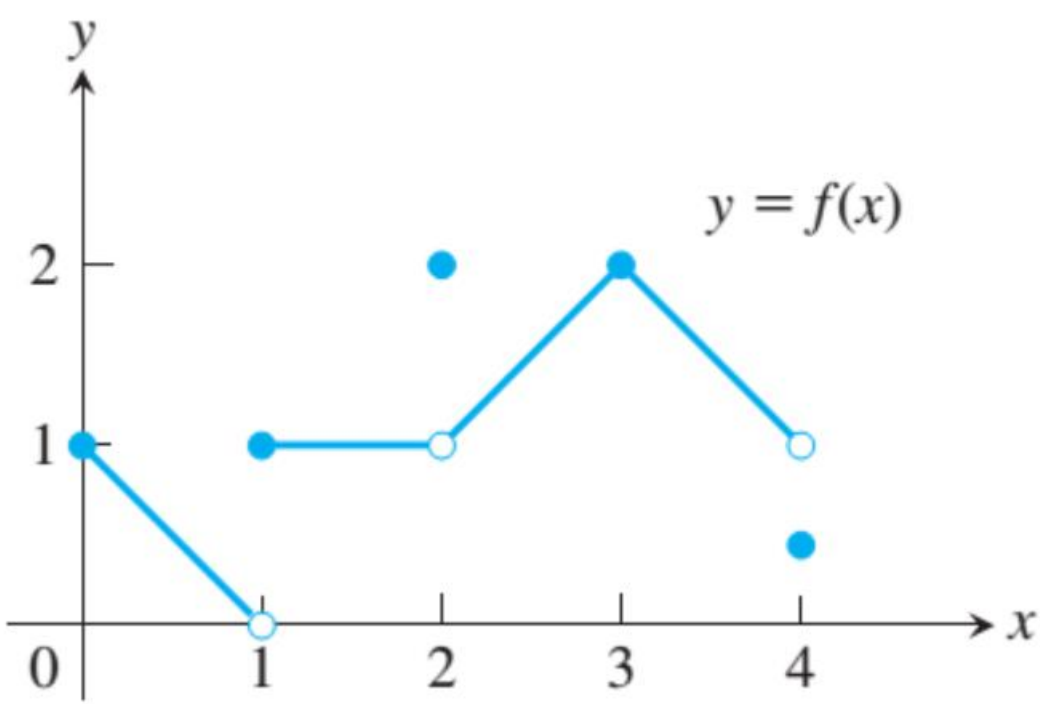
Find

b) $\lim_{x\to 0^-} f(x)$

One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,



Find

b) $\lim_{x \to 0^-} f(x)$

do not exist

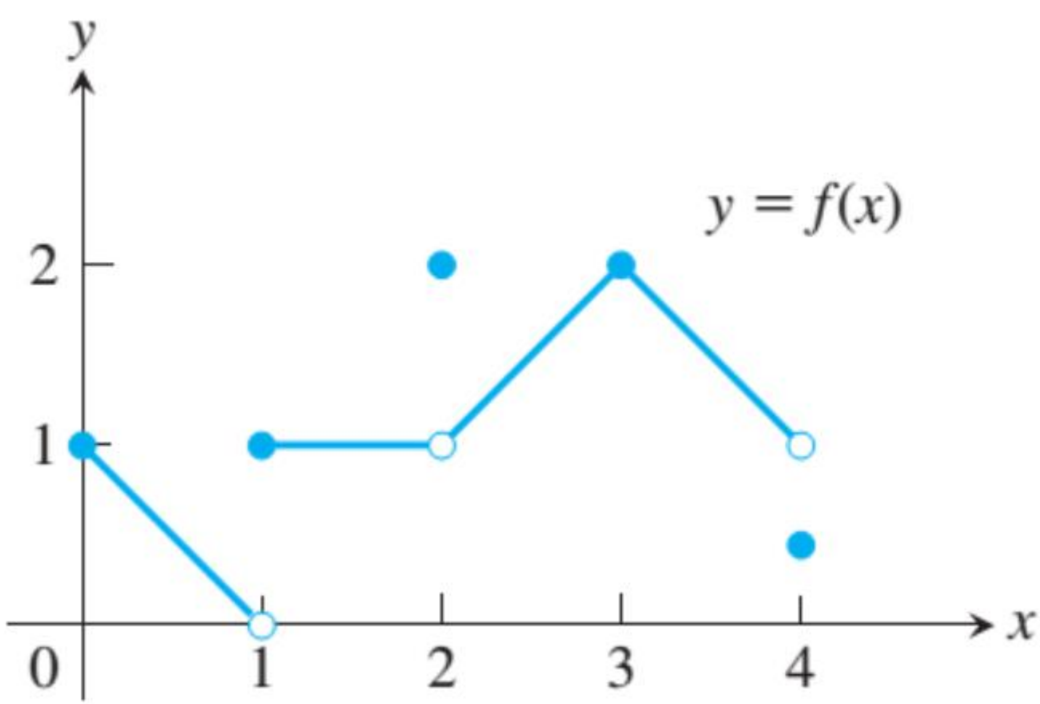
One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,

Find

c) $\lim_{x \to 2^+} f(x)$



One-Sided Limits (6/6)

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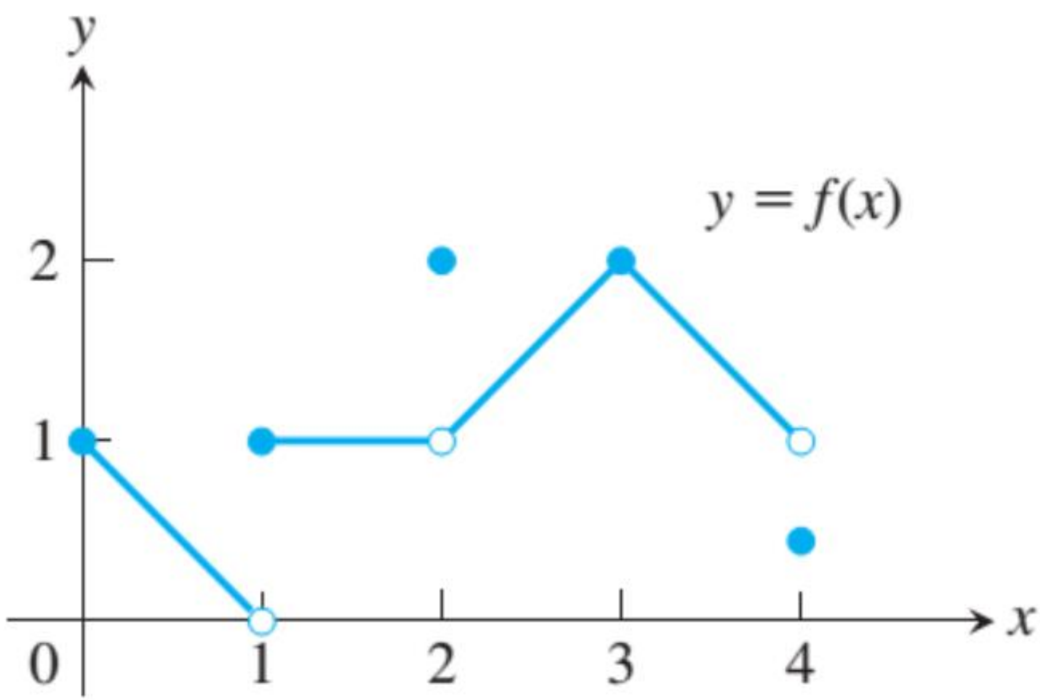
For the function f graphed in the accompanying figure,



<u>Find</u>

c) $\lim_{x \to 2^+} f(x)$

$= 1$

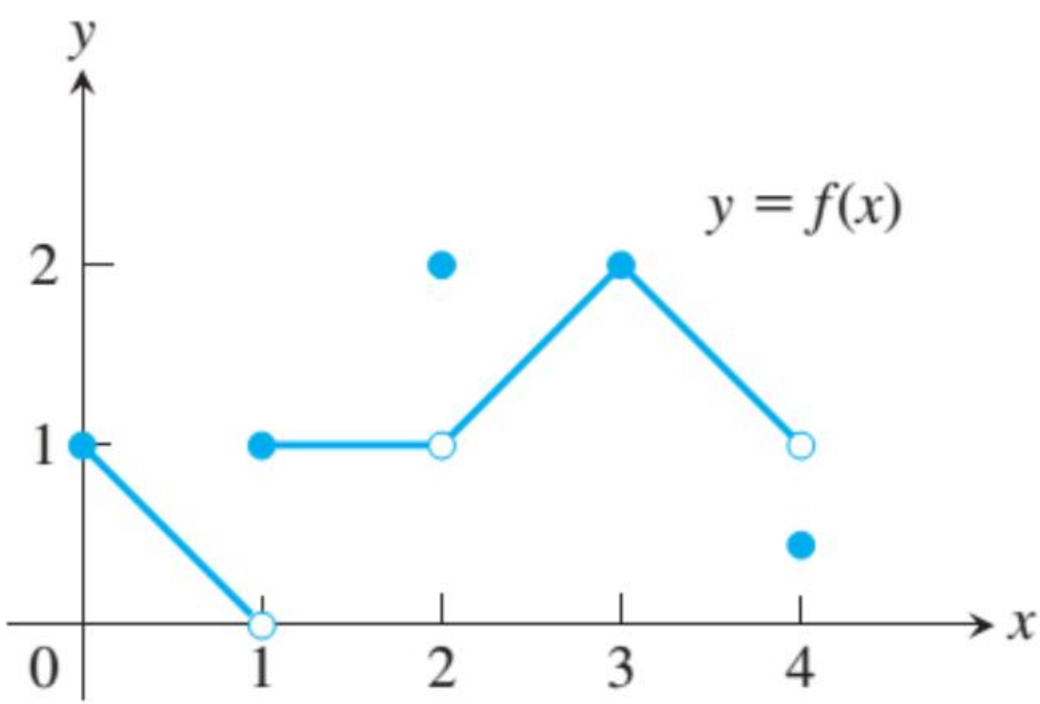
One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,

Find

c) $\lim_{x \to 2^-} f(x)$



One-Sided Limits (6/6)

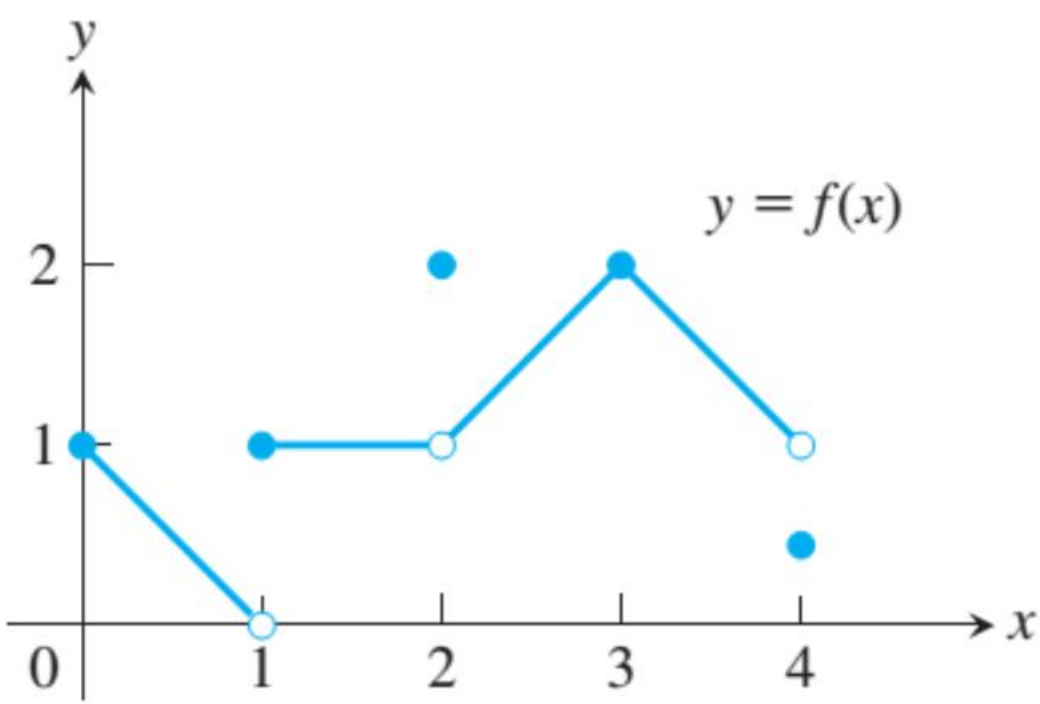
Example 5:

For the function f graphed in the accompanying figure,

Find

c) $\lim_{x \to 2^-} f(x)$

$= 1$



One-Sided Limits (6/6)

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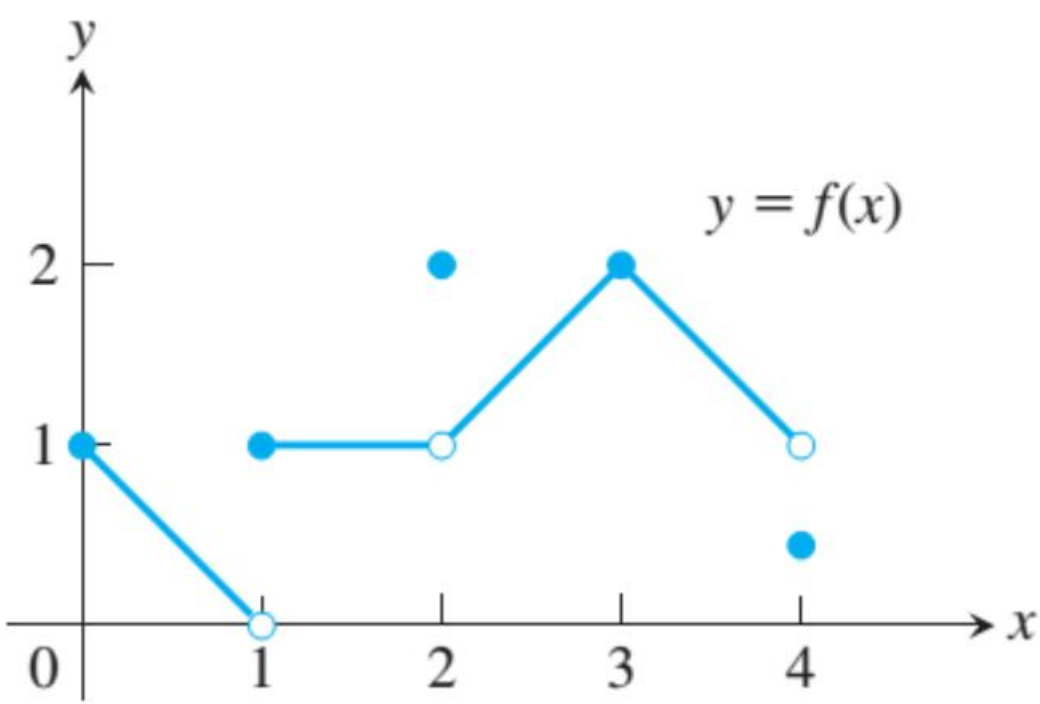


<u>Find</u>

c) $\lim_{x \to 2} f(x)$

One-Sided Limits (6/6)

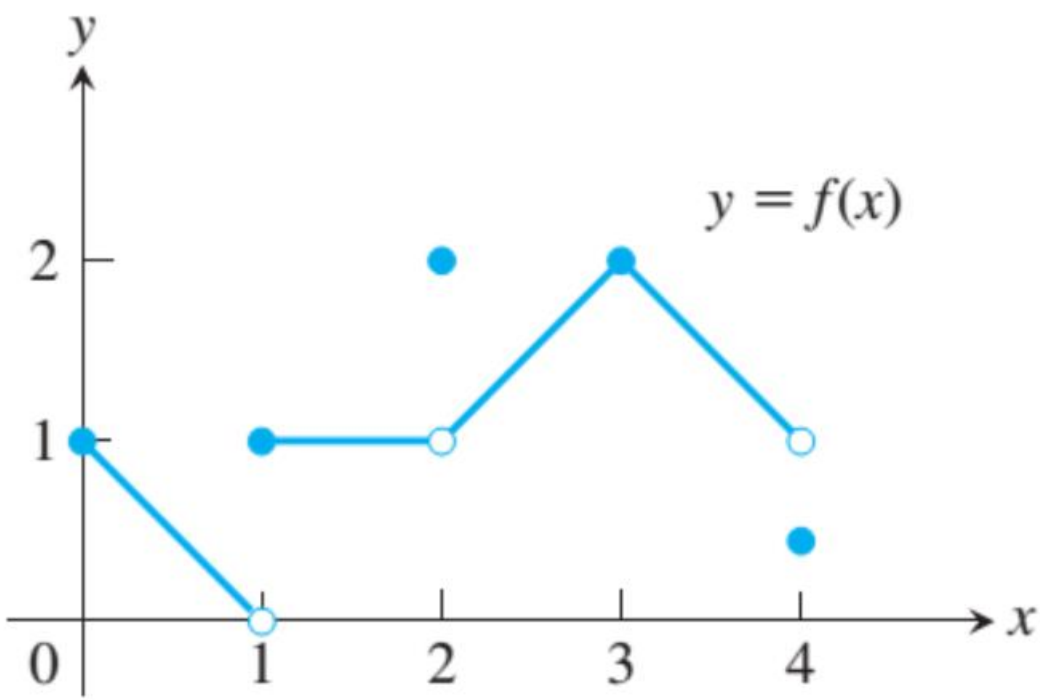
Example 5:

For the function f graphed in the accompanying figure,

Find

c) $\lim_{x \to 2} f(x)$

$= 1$



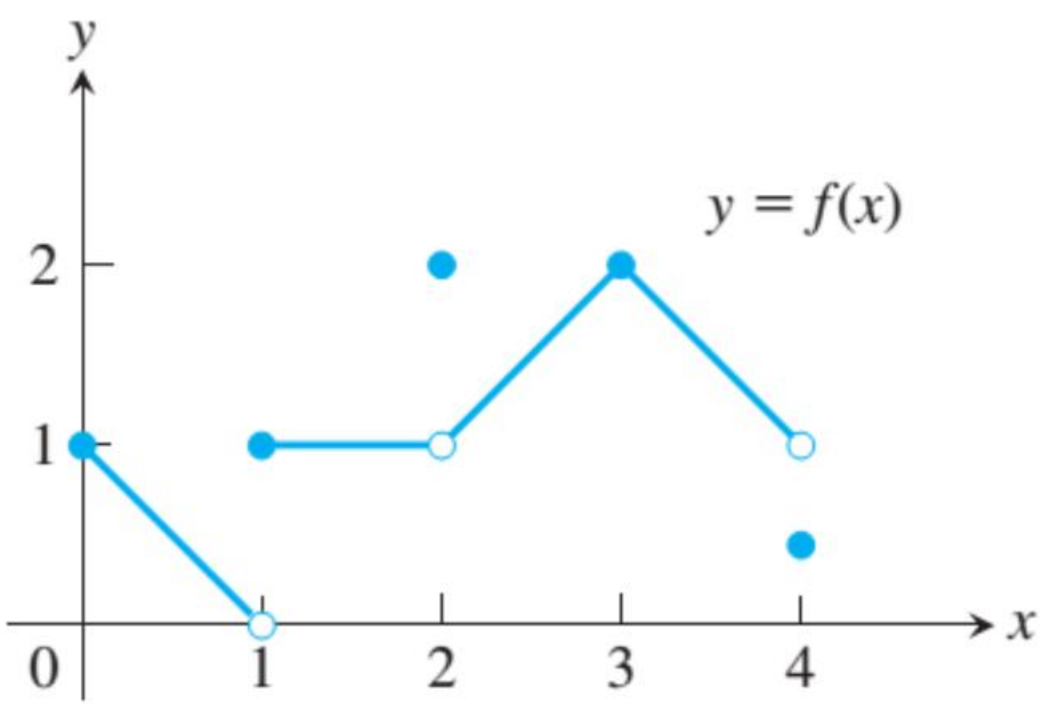
One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,

<u>Find</u>

d) $f(2)$



One-Sided Limits (6/6)

Example 5:

For the function f graphed in the accompanying figure,

Example 5:

For the function f graphed in the accompanying figure,

Find

d) $f(2)$

$= 2$

Find

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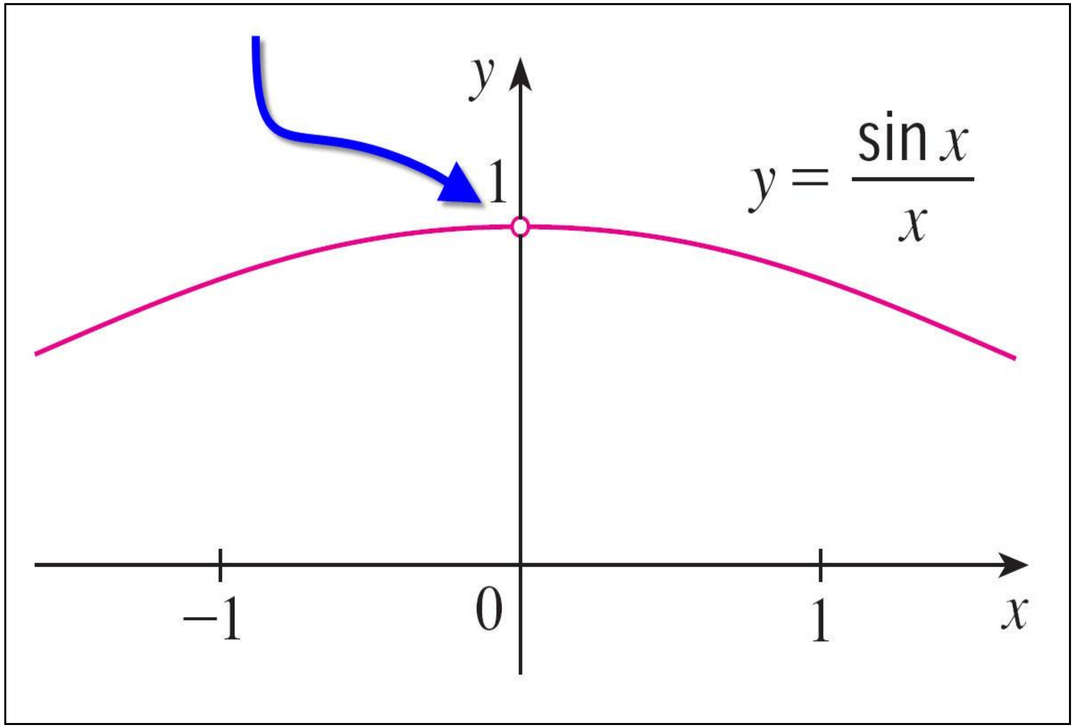
$= 2$

Finding Limit (1/7)

Special Trigonometric Limits (1/3)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

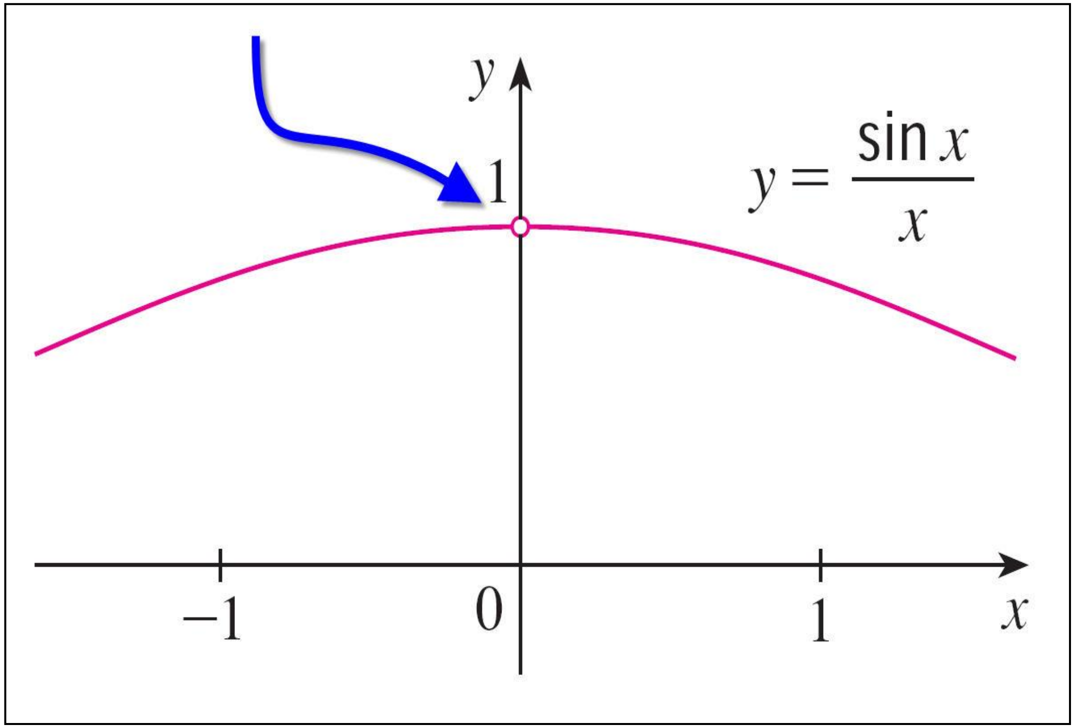


Finding Limit (1/7)

Special Trigonometric Limits (2/3)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\cos x \le \frac{\sin x}{x} \le 1$$



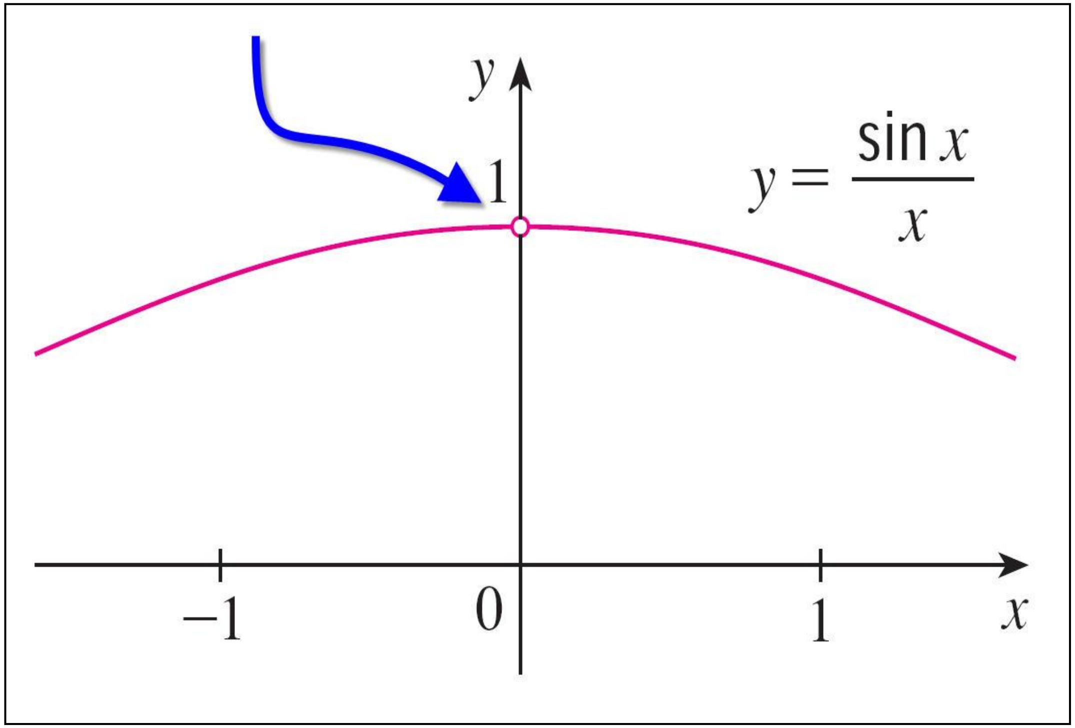
Finding Limit (1/7)

Special Trigonometric Limits (3/3)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

If a, b are constants

$$\lim_{x \to 0} \frac{\sin \boldsymbol{ax}}{\boldsymbol{bx}} = \frac{a}{b}$$



Finding Limit (2/7)

Example 1: Evaluate the following limit (1/2)

$$\lim_{x \to 0} \frac{\sin 5x}{x}$$

If a, b are constants

$$\lim_{x \to 0} \frac{\sin \boldsymbol{ax}}{\boldsymbol{bx}} = \frac{a}{b}$$

Finding Limit (2/7)

Example 1: Evaluate the following limit (2/2)

$$\lim_{x \to 0} \frac{\sin 5x}{x} = 5 \cdot \left(\lim_{x \to 0} \frac{\sin 5x}{5x} \right) = 5 \cdot 1 = 5$$

If a, b are constants

$$\lim_{x \to 0} \frac{\sin \boldsymbol{ax}}{\boldsymbol{bx}} = \frac{a}{b}$$

Finding Limit (3/7)

Example 2: Evaluate the following limit (1/2)

$$\lim_{x \to 0} \frac{\sin x}{5x}$$

If a, b are constants

$$\lim_{x \to 0} \frac{\sin \boldsymbol{ax}}{\boldsymbol{bx}} = \frac{a}{b}$$

Finding Limit (3/7)

Example 2: Evaluate the following limit (2/2)

$$\lim_{x\to 0}\frac{\sin x}{5x}=\frac{1}{5}\cdot\left(\lim_{x\to 0}\frac{\sin x}{x}\right)=\frac{1}{5}\cdot 1=\frac{1}{5}$$

If a, b are constants

$$\lim_{x\to 0}\frac{\sin \boldsymbol{ax}}{\boldsymbol{bx}}=\frac{a}{b}$$

Finding Limit (4/7)

Special Trigonometric Limits

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$= \lim_{x\to 0} \frac{\sin x}{x\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x}$$

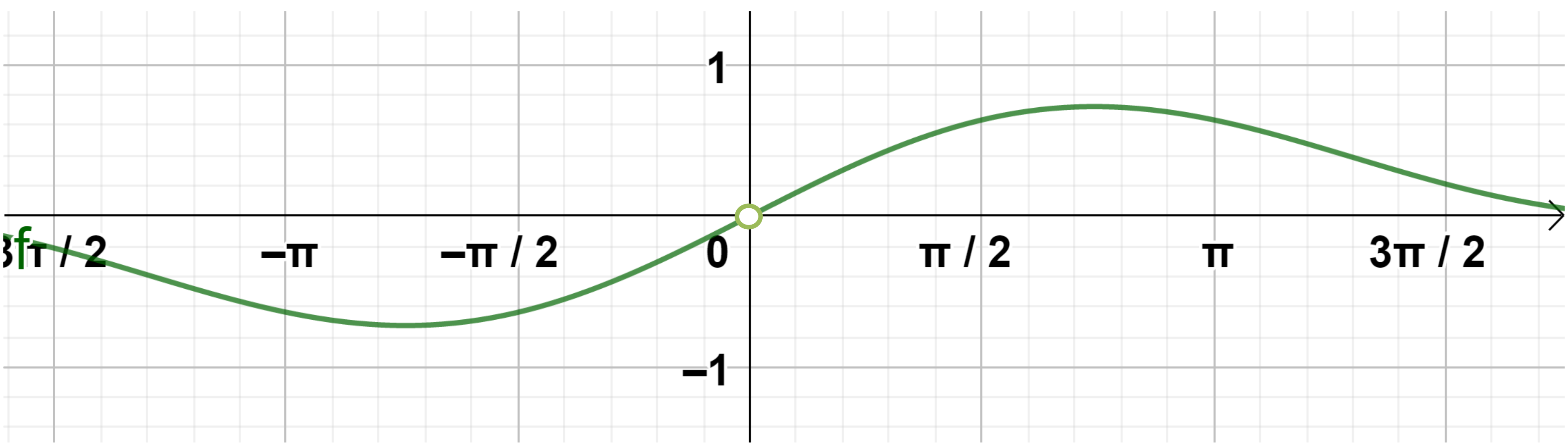
$$= 1 \cdot 1 = 1$$

Finding Limit (5/7)

Special Trigonometric Limits

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Multiplying by the conjugate



Finding Limit (6/7)

Limit doesn't exist

Oscillating Behavior (1/3):

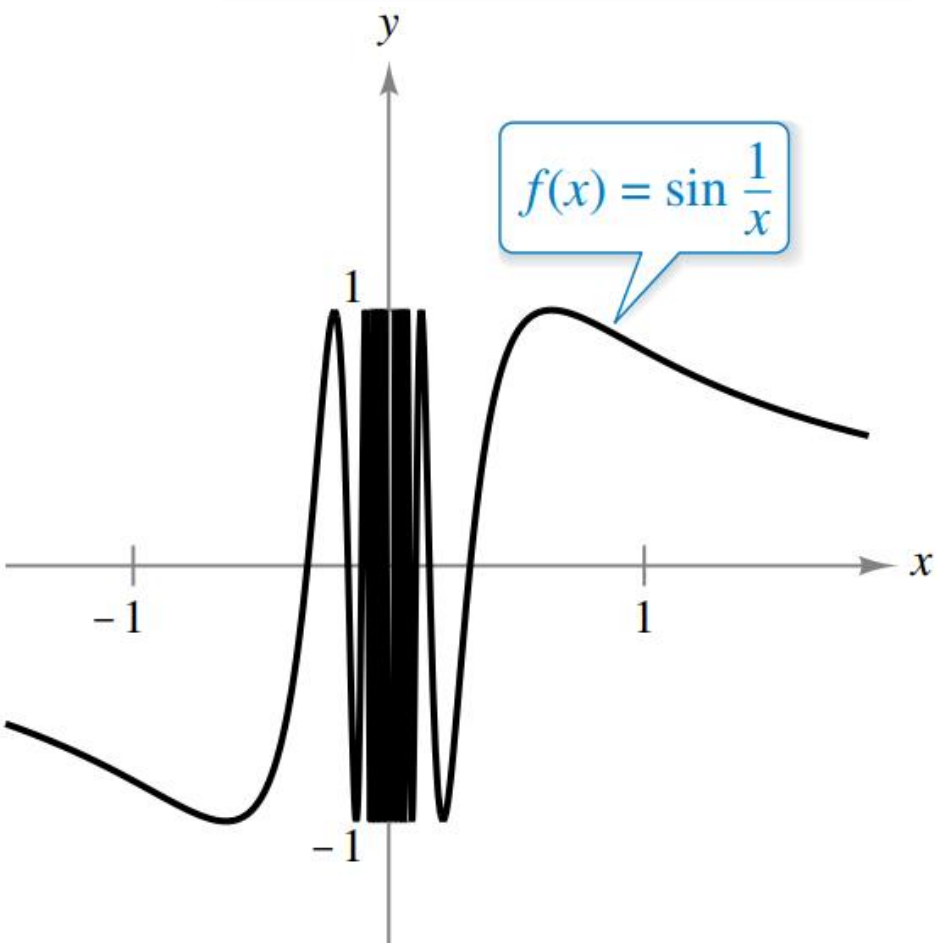
Find $\lim_{x \to 0} \sin\frac{1}{x}$ if it exists.

Finding Limit (6/7)

Oscillating Behavior (2/3):

Limit doesn't exist

Find $\lim_{x \to 0} \sin \frac{1}{x}$ if it exists.



$\lim_{x \to 0} \sin \frac{1}{x}$ doesn't exist:

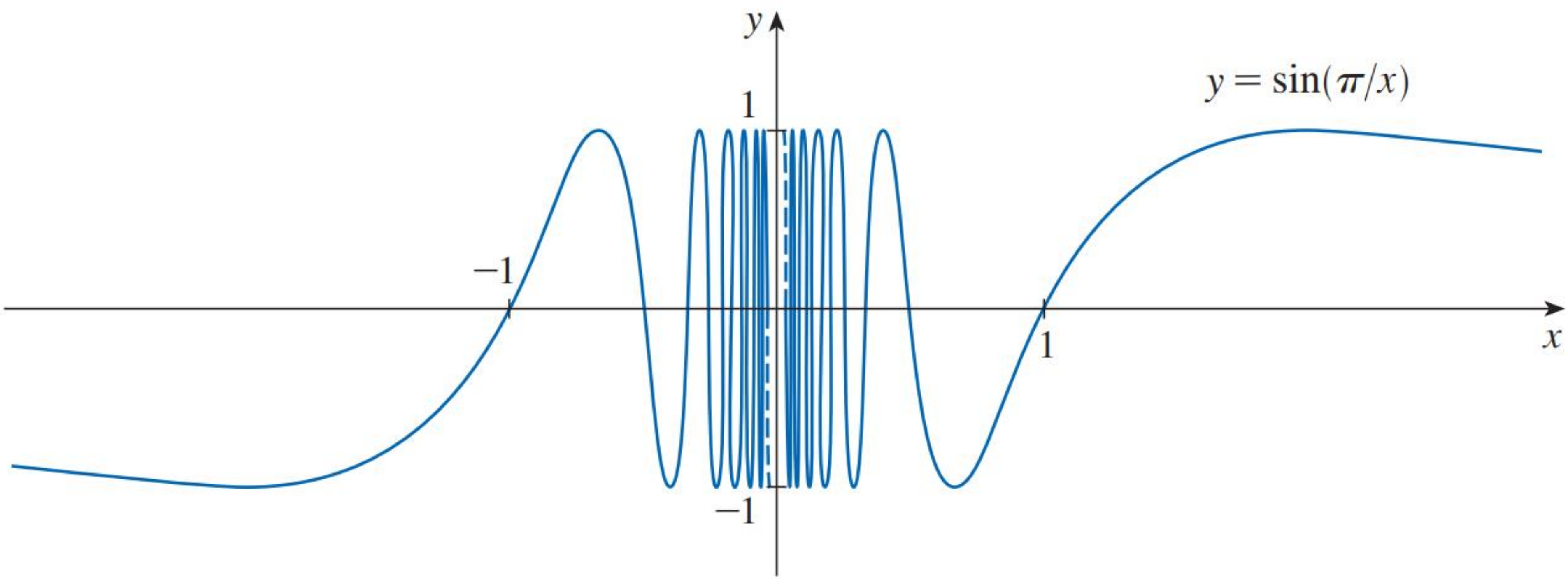
$f(x)$ oscillates between two fixed values $\{1, -1\}$ as x approaches 0.

Finding Limit (6/7)

Oscillating Behavior (3/3):

Oscillate between 1 and -1 infinitely often as *x* approaches 0

$$\lim_{x \to 0} \sin(\pi/x) \text{ does not exist.}$$



Finding Limit (7/7)

Limit doesn’t exist

Unbounded Behavior (1/4):

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

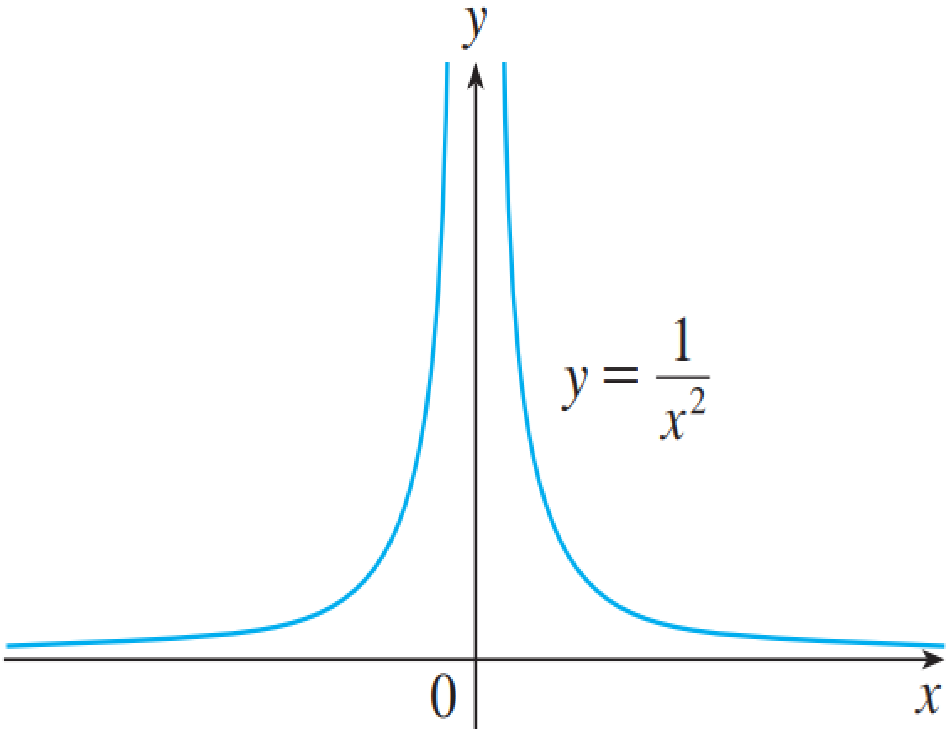
Finding Limit (7/7)

Unbounded Behavior (2/4):

Limit doesn't exist

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

x	$\frac{1}{x^2}$
±1	1
±0.5	4
±0.2	25
±0.1	100
±0.05	400
±0.01	10,000
±0.001	1,000,000



Finding Limit (7/7)

Unbounded Behavior (3/4):

Limit doesn't exist

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

You can see that as x approaches 0 from either the right or the left, $f(x)$ increases without bound.

Because $f(x)$ does not become arbitrarily close to a single number L as x approaches 0, you can conclude that the **limit does not exist**.

Finding Limit (7/7)

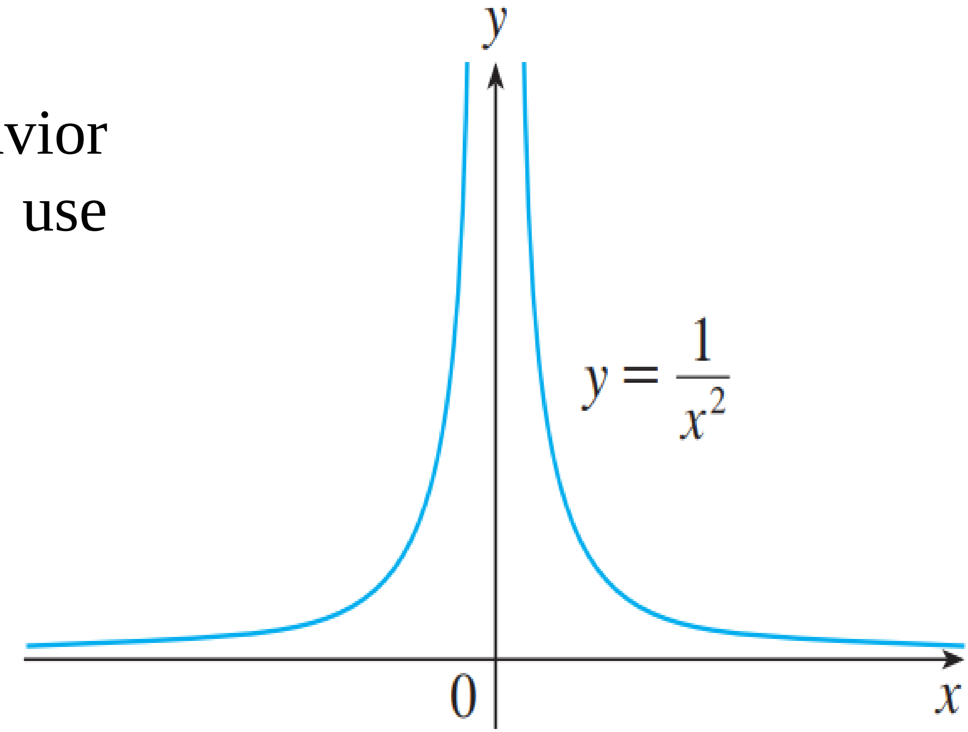
Unbounded Behavior (4/4):

Limit doesn't exist

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

To indicate the kind of behavior exhibited in this example, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

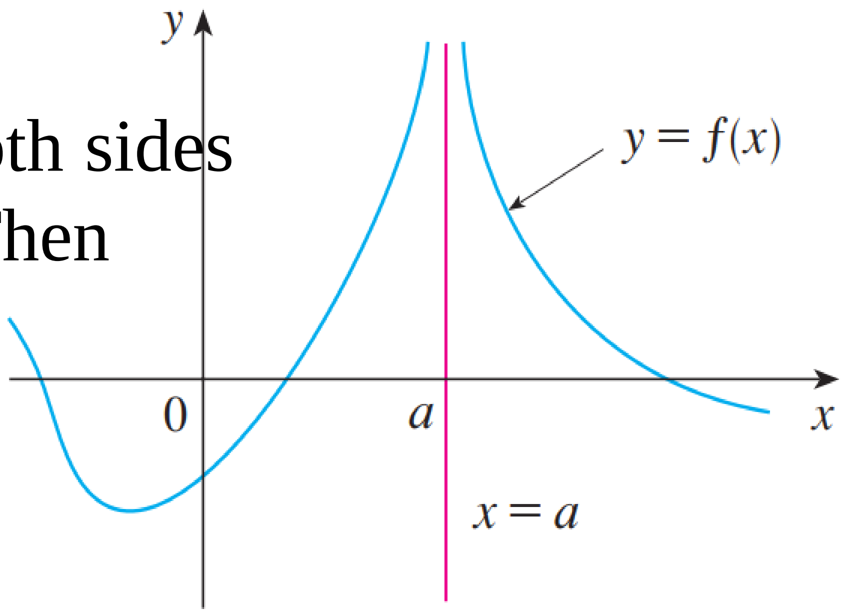


Infinite Limits (1/6)

Definition (1/2):

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$



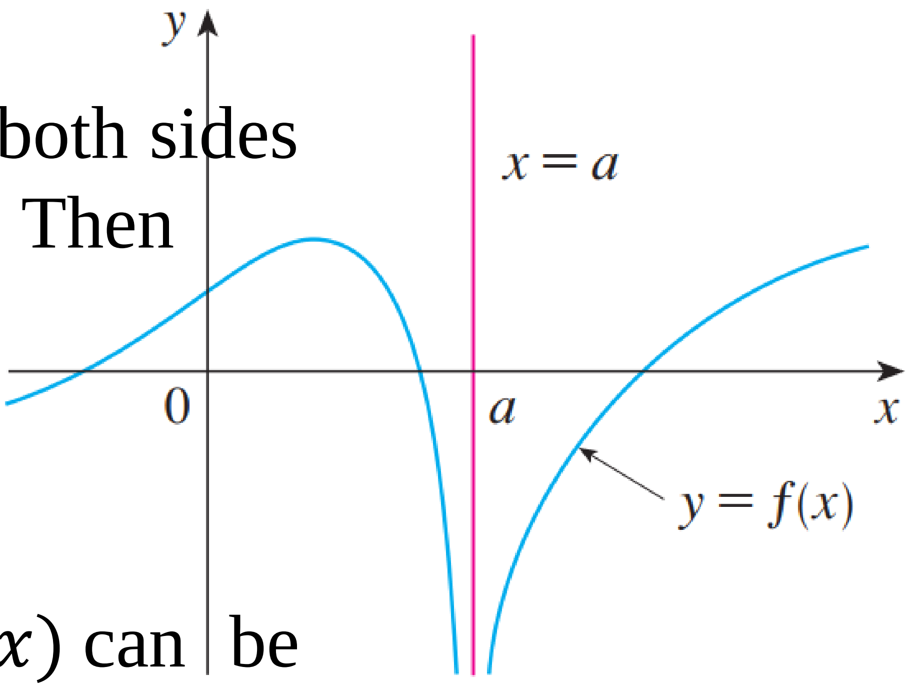
means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Infinite Limits (1/6)

Definition (2/2):

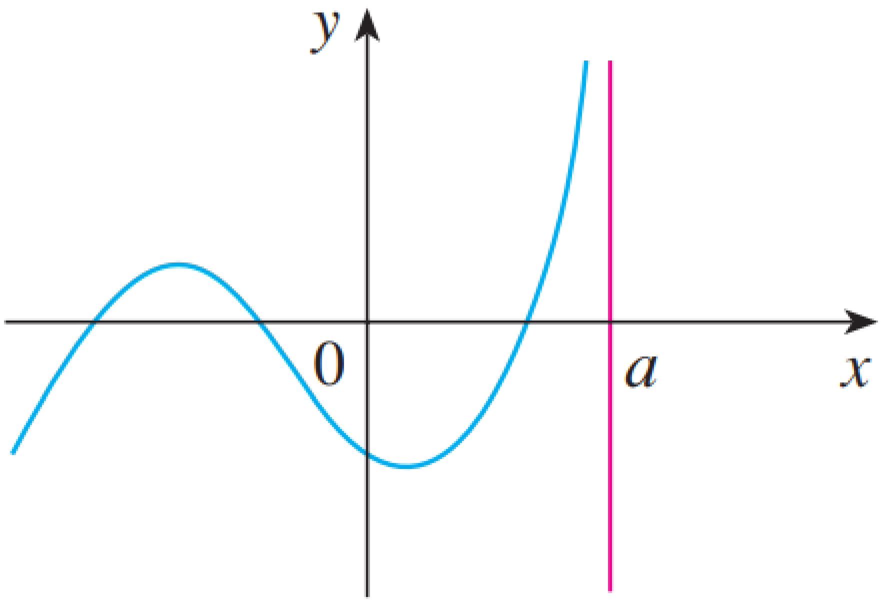
Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

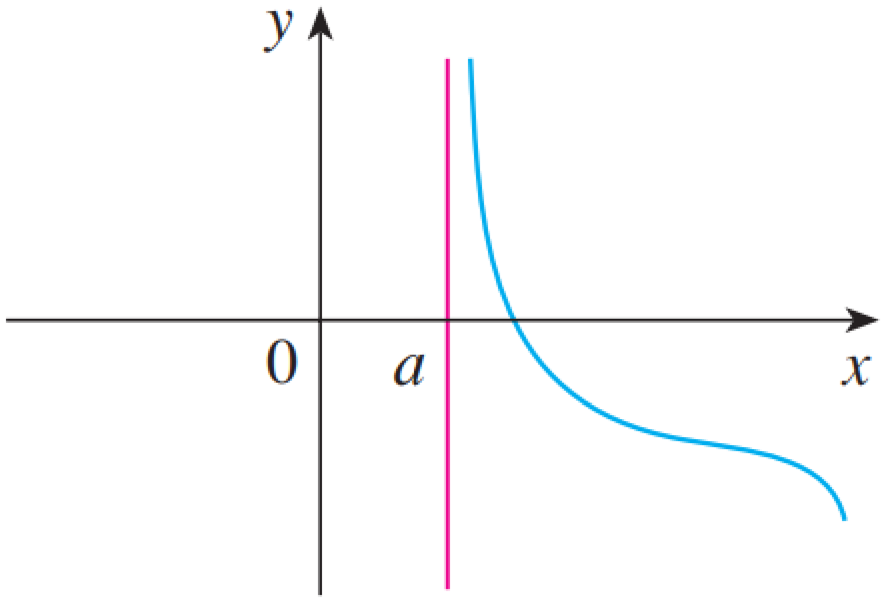


means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

One-Sided (1/2):



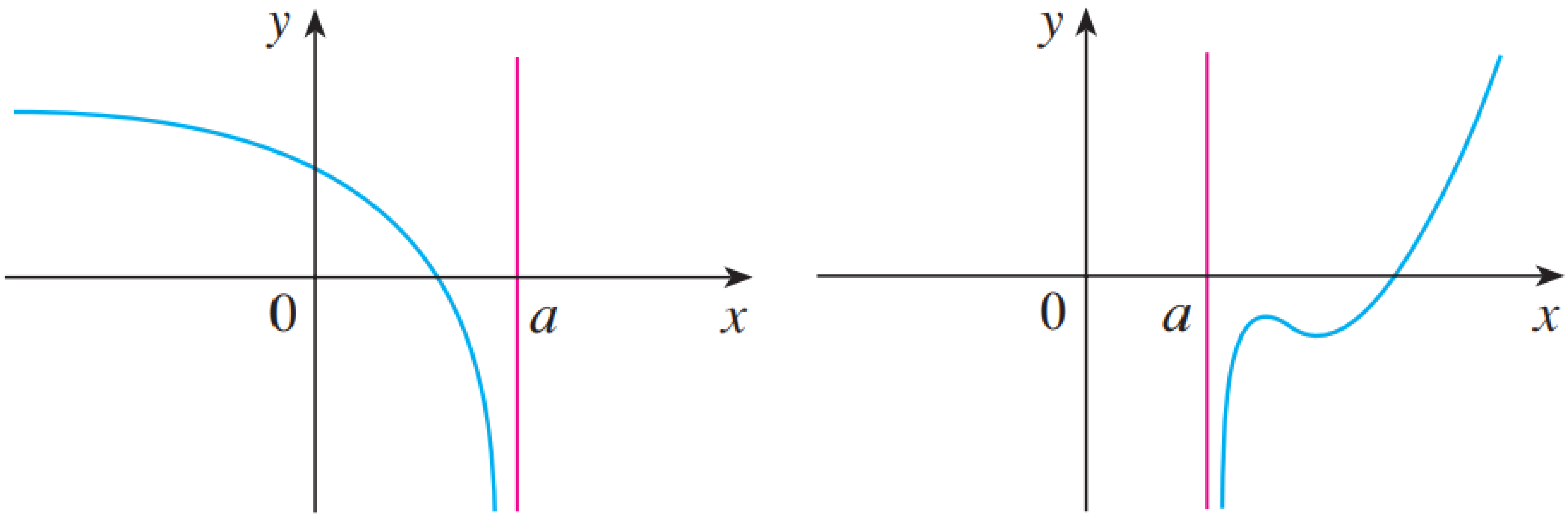
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$

Infinite Limits (2/6)

One-Sided (2/2):



(c) $\lim_{x \to a^-} f(x) = -\infty$

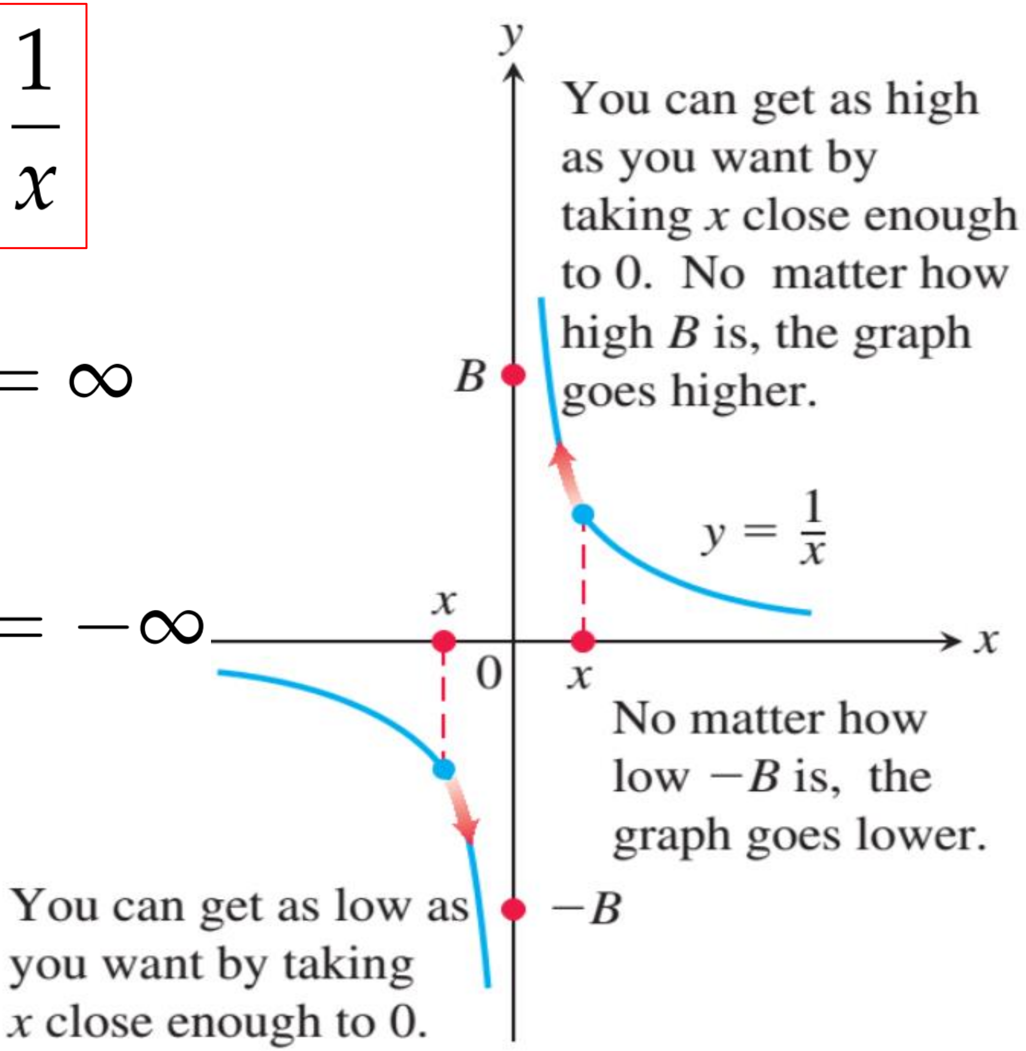
(d) $\lim_{x \to a^+} f(x) = -\infty$

Infinite Limits (3/6)

Example 1: $$f(x) = \frac{1}{x}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

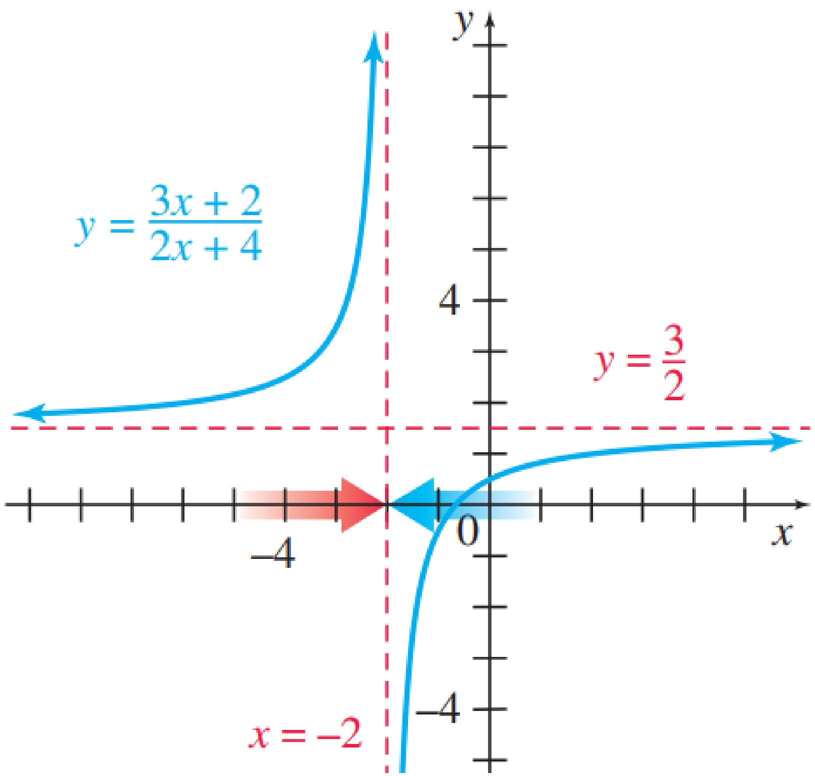


Infinite Limits (4/6)

Example 1:

Find $\lim_{x \to -2} f(x)$, where

$$f(x) = \frac{3x + 2}{2x + 4}$$



x approaches −2 from left

x approaches −2 from right

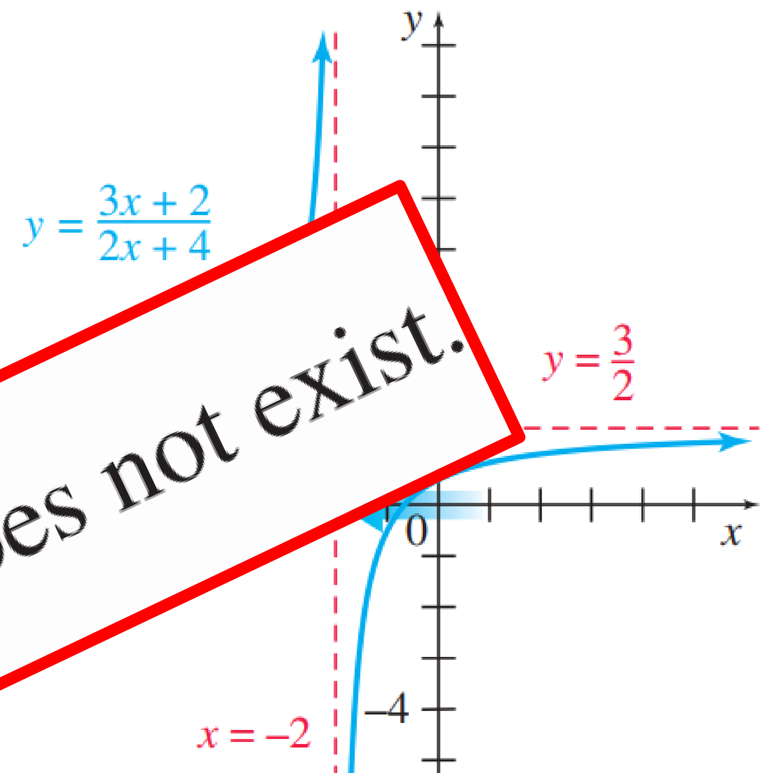
x	−2.1	−2.01	−2.001	−2.0001	−1.9999	−1.999	−1.99	−1.9
$f(x)$	21.5	201.5	2001.5	20,001.5	−19,998.5	−1998.5	−198.5	−18.5

Infinite Limits (4/6)

Example 1:

Find $\lim_{x \to -2} f(x)$, where

$$f(x) = \frac{3x + 2}{2x + 4}$$



$$\lim_{x \to -2} \frac{3x + 2}{2x + 4} \text{ does not exist.}$$

rom left

x approaches -2 from right

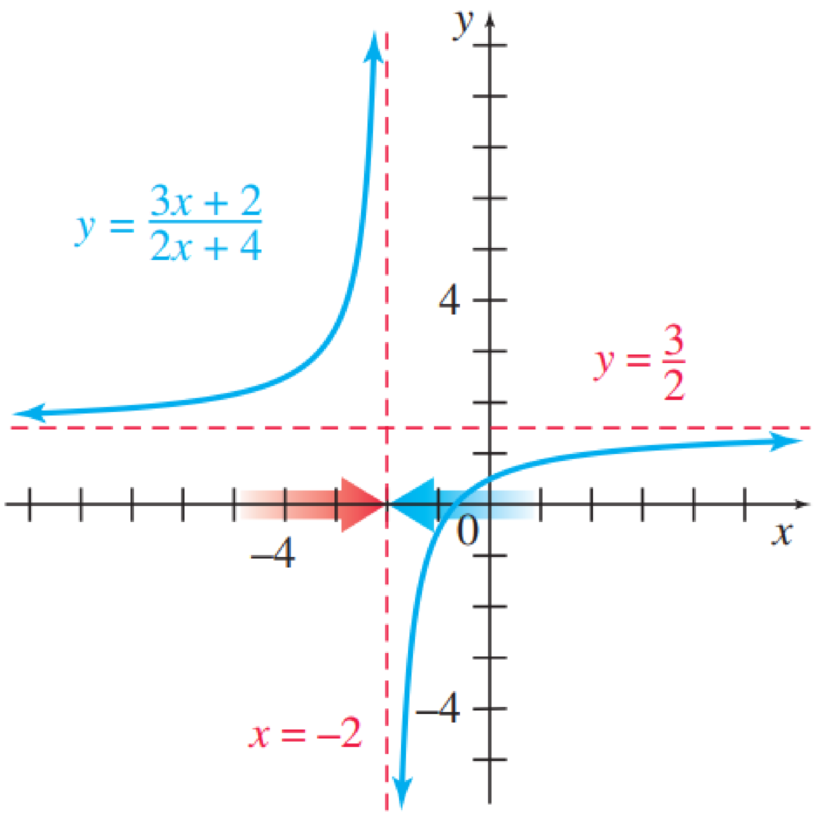
x	-2.1	[illegible]	-2.001	-2.0001	-1.9999	-1.999	-1.99	-1.9
$f(x)$	21.5	201.5	2001.5	20,001.5	−19,998.5	−1998.5	−198.5	−18.5

Infinite Limits (4/6)

Example 1:

Find $\lim_{x \to -2} f(x)$, where

$$f(x) = \frac{3x + 2}{2x + 4}$$



$$\lim_{x \to -2^-} f(x) = \infty.$$

$$\lim_{x \to -2^+} f(x) = -\infty,$$

$$\lim_{x \to -2} \frac{3x + 2}{2x + 4} \text{ does not exist.}$$

Infinite Limits (5/6)

Example 2:

Find $\lim_{x \to \pi/2} \tan x$.

Infinite Limits (5/6)

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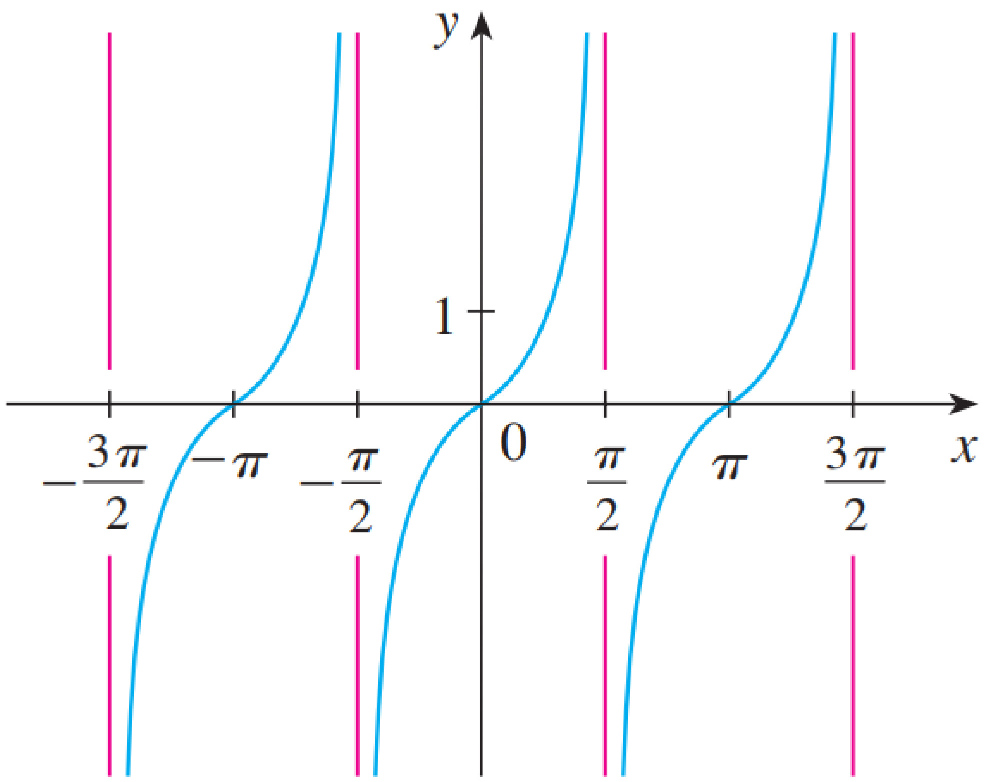


Infinite Limits (5/6)

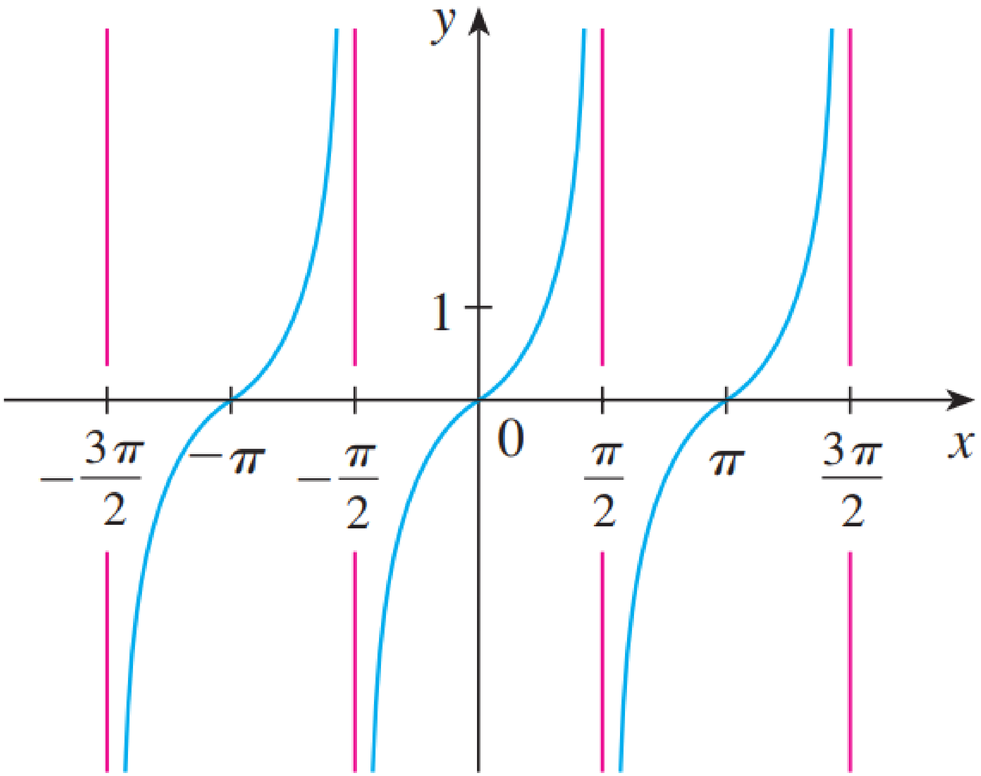
Example 2:

Find $\lim_{x\to\pi/2} \tan x$.

$\lim_{x\to\pi/2^-} \tan x = +\infty$ and

$\lim_{x\to\pi/2^+} \tan x = -\infty$ then

$\lim_{x\to\pi/2} \tan x$ is doesn't exist



Infinite Limits (6/6)

Example 3:

Find $\lim_{x \to 0^+} \ln x$.

Infinite Limits (6/6)

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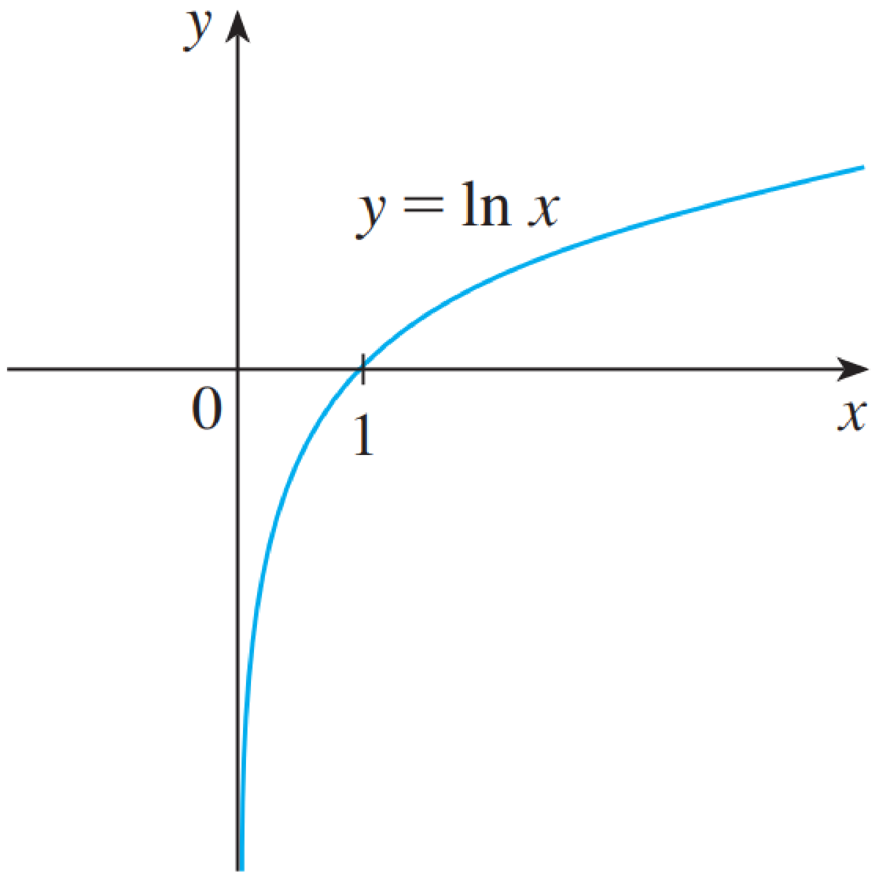
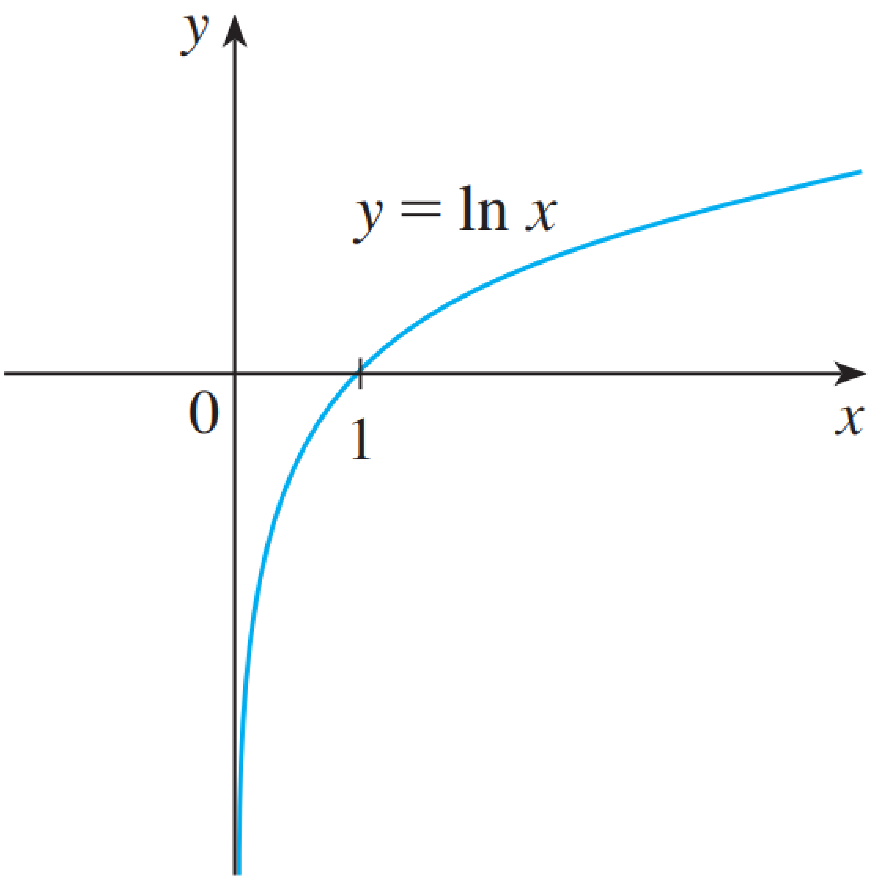


Infinite Limits (6/6)

Example 3:

Find $\lim_{x \to 0^+} \ln x$.

$$\lim_{x \to 0^+} \ln x = -\infty$$

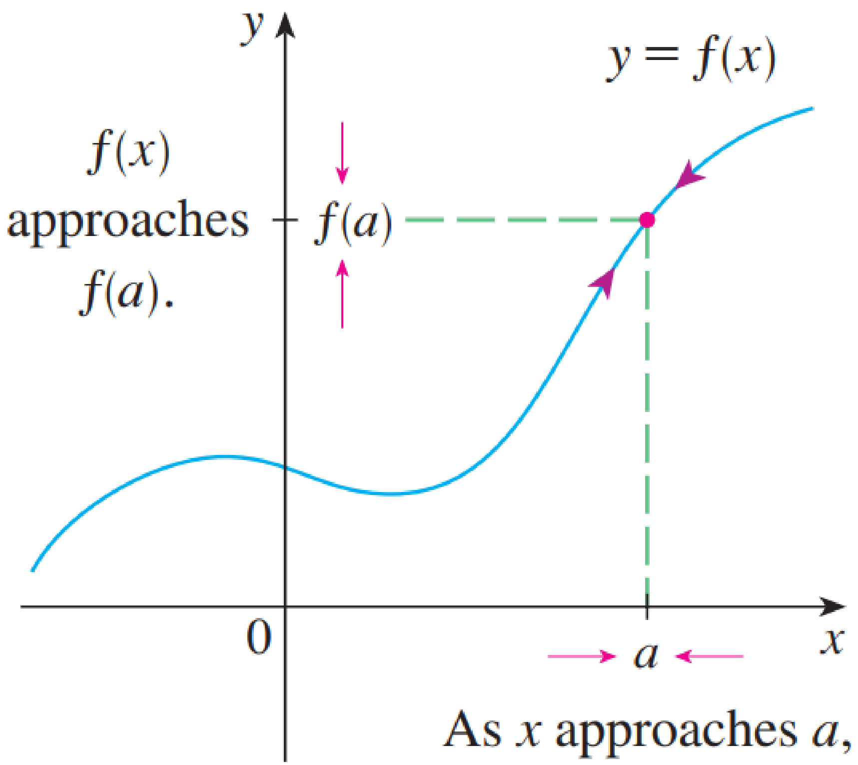


Continuity (1/5)

Definition:

A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$



Continuity (2/5)

Definition:

A function f is **continuous at a number a** if

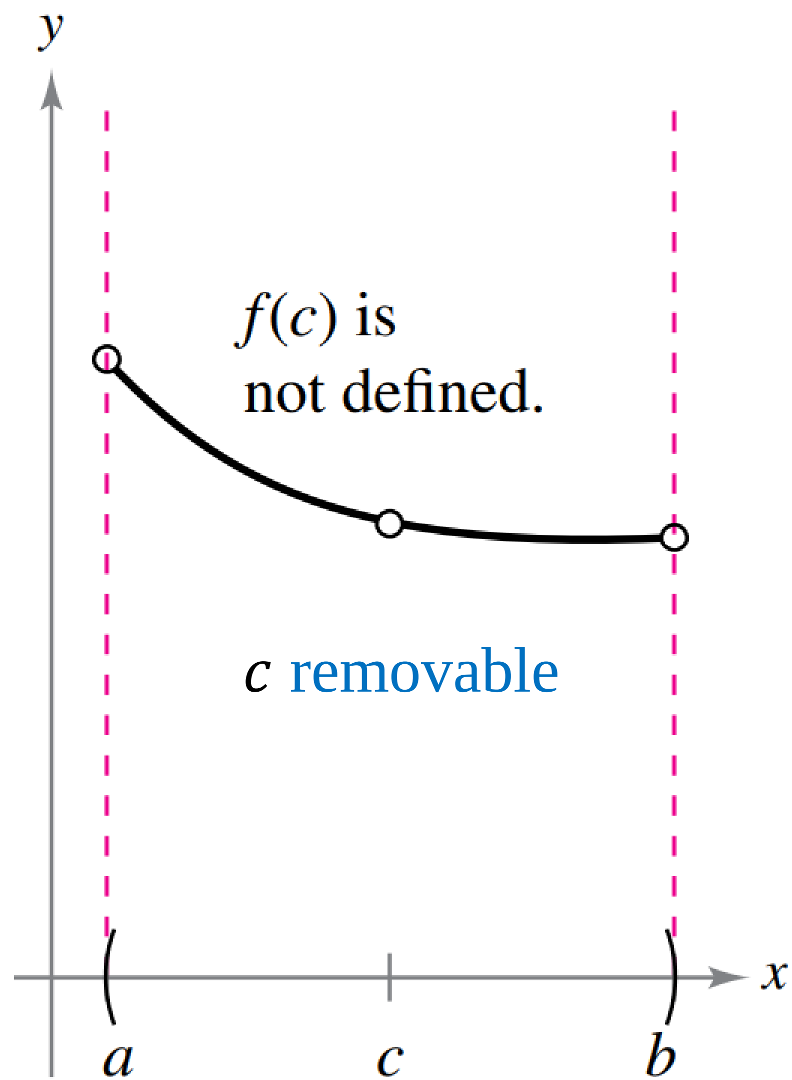
$$\lim_{x \to a} f(x) = f(a)$$

1. $f(a)$ is defined (that is, a is in the domain of f),
2. $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$.

Continuity (3/5)

f is **not** continuous at $x = c$

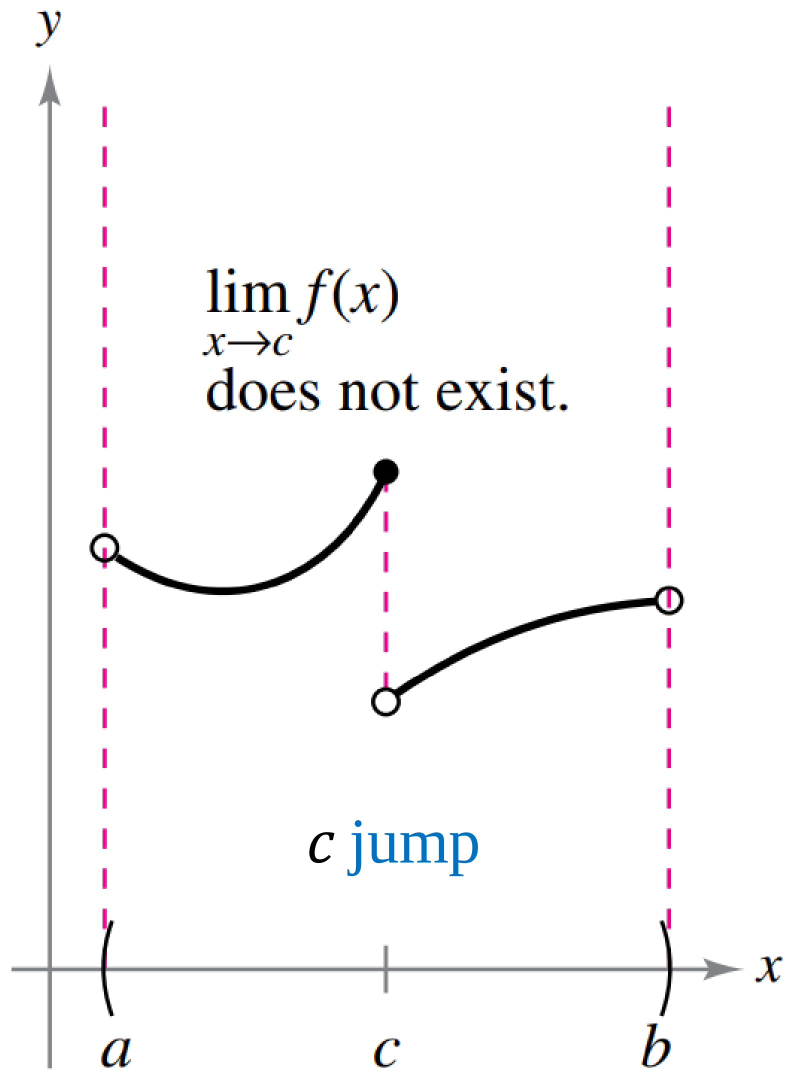
1



Continuity (3/5)

f is **not** continuous at $x = c$

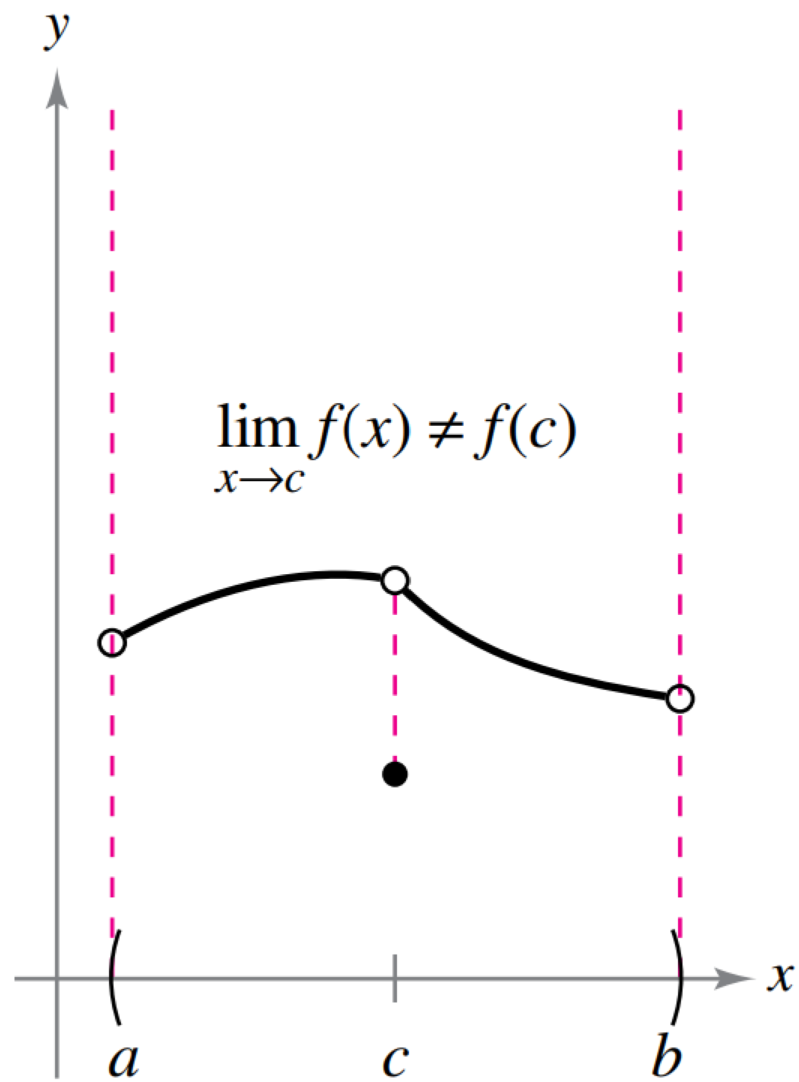
2



Continuity (3/5)

f is **not** continuous at $x = c$

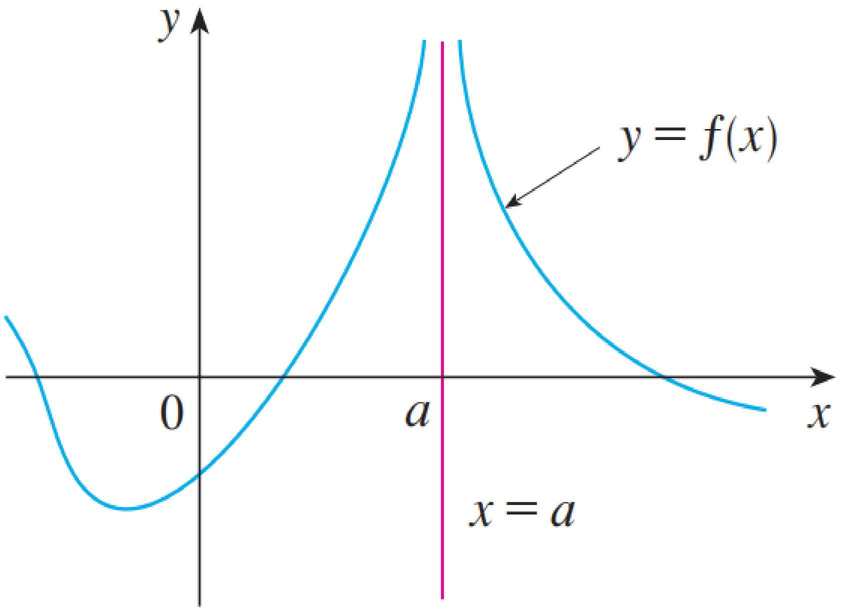
3



Continuity (3/5)

f is **not** continuous at $x = a$

4



If the graph of a function f has a vertical asymptote at $x = a$, then f is not continuous at a.

a infinte

Continuity (4/5)

Example 1:

Is the following functions discontinuous?

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

Continuity (4/5)

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Notice that $f(2)$ is not defined,

so f is discontinuous at $x = 2$.

Continuity (4/5)

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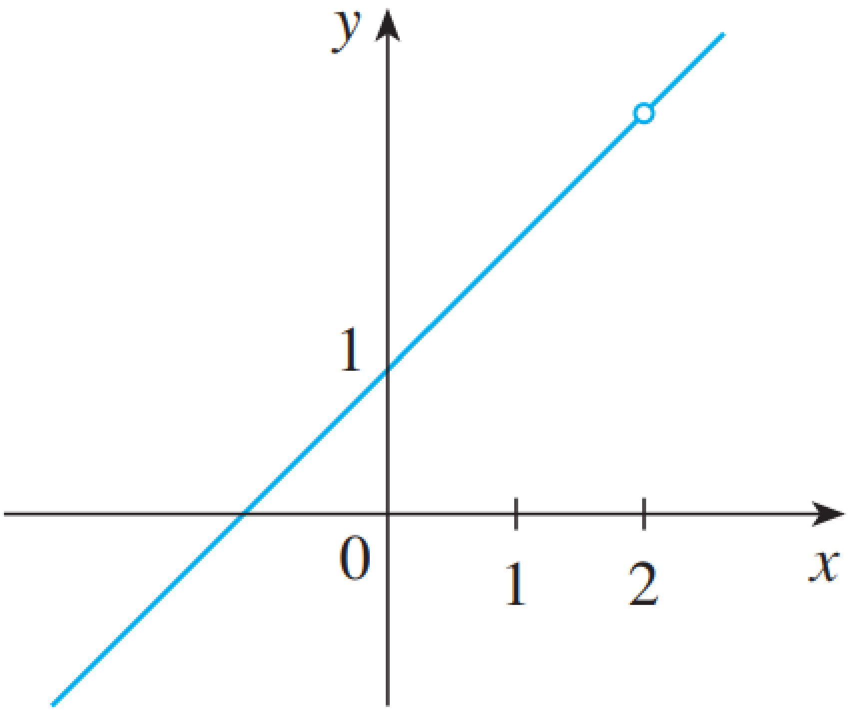

Notice that $f(2)$ is not defined,

so f is discontinuous at $x = 2$.

Continuity (5/5)

Example 2:

Is the following functions discontinuous?

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Continuity (5/5)

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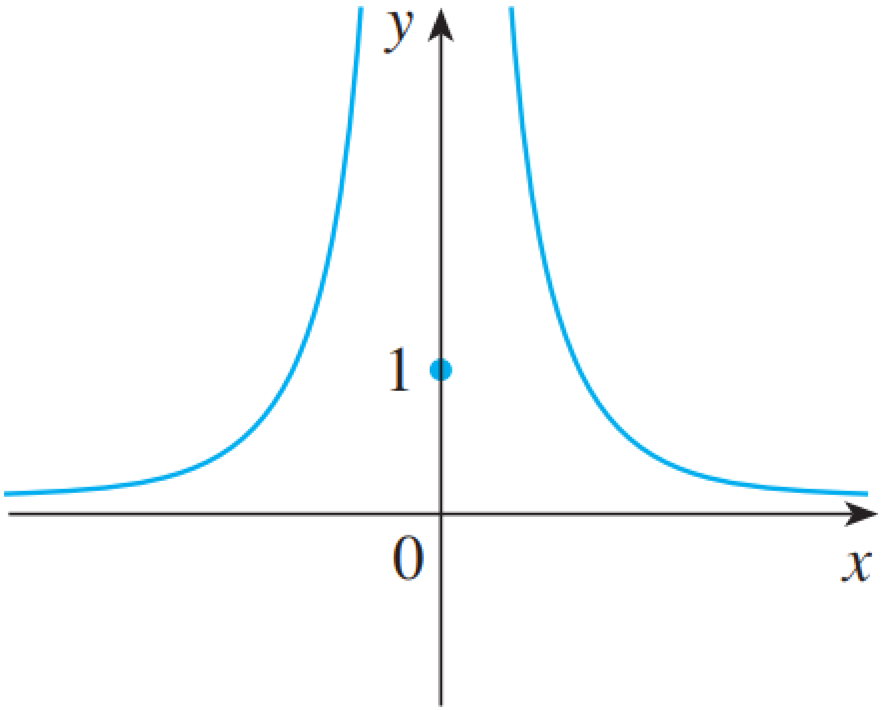

Continuity (5/5)

Example 2:

Is the following functions discontinuous?

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

f is discontinuous at $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} \quad \text{does not exist.}$$

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gkSI_PPAVJpebKDLo-ijEC

Lecture #4: https://www.youtube.com/watch?v=yywfUe84z6E&list=PLxIvc-MGOs6gkSI_PPAVJpebKDLo-ijEC&index=5

Thank You

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